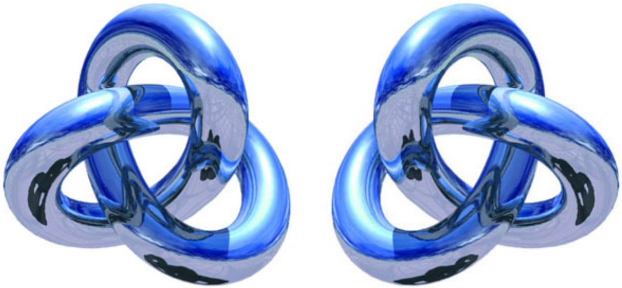
Chirality in Dynamics



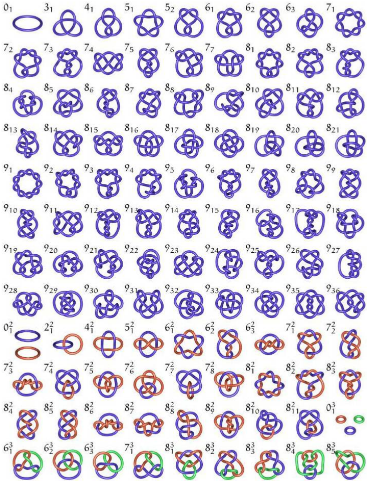
Étienne Ghys

CNRS – École Normale Supérieure de Lyon

Rio de Janeiro, September 2009

"I call any geometrical figure, or group of points, chiral, and say it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself." (Kelvin, 1904)

Chirality in Topology



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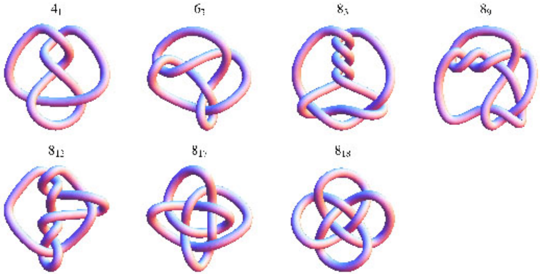

There are 20 amphichiral (= non chiral) among the 165 prime knots having ten or fewer crossings.

A very simple chiral dynamical system

Two independent oscillators : dynamics on $\mathbf{C}^2 \simeq \mathbf{R}^4$:

$$\phi^t(z_1, z_2) = (\exp(it)z_1, \exp(it)z_2)$$

• $|z_1|^2 + |z_2|^2$ is invariant, so the dynamics is on hyper-spheres $\mathbf{S}^3$ in $\mathbf{R}^4$.

• Orbits are the fibers of the Hopf fibration :

$$(z_1, z_2) \in \mathbf{S}^3 \subset \mathbf{C}^2 \mapsto \frac{z_1}{z_2} \in \mathbf{C} \cup \{\infty\} \simeq \mathbf{S}^2.$$

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The Hopf fibration

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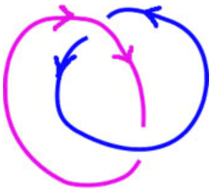
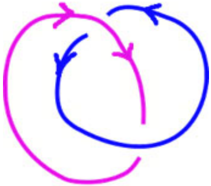
More complicated : The Lorenz attractor

Back to topology : linking numbers



Linking $= +1$

Back to topology : linking numbers



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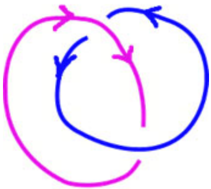

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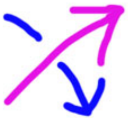

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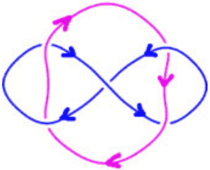

Linking $= 0$

Positivity

A knot (or a link) is called positive if it can be represented in such a way that all crossings are positive.

Fact : Positive knots are chiral.

Classical topological invariant called *signature* which :
1) changes sign in a mirror.
2) is > 0 for positive knots.

Example : the signature is $+2$ for the right handed trefoil knot and -2 for the left handed trefoil.

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Theorem (Murasugi) : Positive knots are fibered.

There is a fibration from the complement $\mathbf{S}^3 \setminus k$ of the knot k onto the circle $\mathbf{S}^1$.

The 3-sphere has an "open book decomposition". It is filled by surfaces (the *pages*), all having the knot as boundary (the *binding of the book*).

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The trefoil knot as a fibered knot

Hopf links are fibered

$$(z_1, z_2) \in \mathbf{S}^3 \backslash (\{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 = z_2\}) \mapsto Arg\,(z_1.z_2.(z_1 - z_2)) \in \mathbf{S}^1.$$

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Positivity in dynamics

Definition : A vector field X on the 3-sphere, generating a dynamical system ϕ_t, is called right handed if for any two distinct points x_1, x_2 the trajectories starting from x_1 and x_2 link positively :

$$\liminf_{T_1,T_2 \to \infty} \frac{1}{T_1 T_2} \mathit{linking}\left([x_1, \phi_{T_1}(x_1)], [x_2, \phi_{T_2}(x_2)]\right) > 0.$$

Examples :

Hopf flow,

Stable property : Small perturbations of right handed vector fields are right handed. Example : weakly coupled oscillators, geodesics on a small perturbation of the round sphere.

Lorenz flow.

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Characterization

Let $\mathcal{P}$ be the compact convex set of probability measures which are invariant under the flow (for instance, periodic orbits).

There is a well defined quadratic linking form:

$$link : (\mu_1, \mu_2) \in \mathcal{P} \times \mathcal{P} \mapsto linking(\mu_1, \mu_2) \in \mathbf{R}$$

Theorem: *A vector field X on the 3-sphere is right handed if and only if the linking quadratic form is positive.*

Right handed vector fields have positive helicity and signature.

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$$linking(x(t), y(s)) = \int\int \frac{\det(\frac{dx}{dt}, \frac{dy}{ds}, x(t) - y(s))}{||x(t) - y(s)||^3}\, dtds$$

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Theorem : *Right-handed vector fields are "fibered".*

- *Periodic orbits are fibered knots.*
- *The complement of the support of any invariant measure carries a non singular closed form.*
- *Any finite collection of periodic orbits is the binding of some Birkhoff section, an open book transverse to the vector field.*

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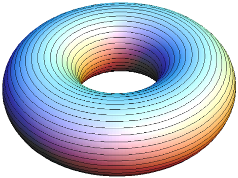
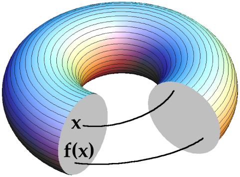
Some words about proofs.

Analogous to the classical problem in dynamics :

Given a vector field in some (compact) manifold, does there exist a global Poincaré section ?





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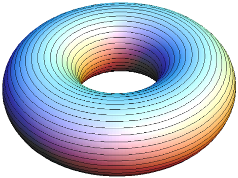

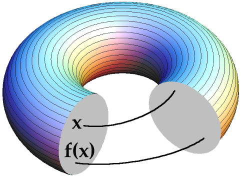

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Let $\gamma(t)$ be a periodic orbit.

$$\omega_x(v) = \int \Omega_{\gamma(t),x}(\frac{d\gamma}{dt}, v)\, dt$$

is a closed form.

Its "primitive" is a map :

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The Lorenz attractor is fibered

Two helices in Chambord