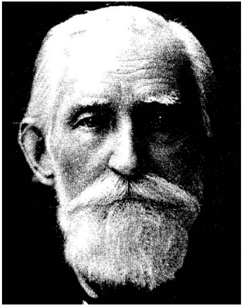
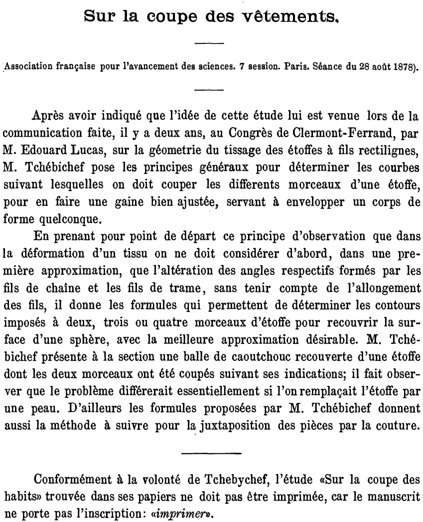
From Chebyshev nets
to
Painlevé differential equations



Unité de Mathématiques Pures et Appliquées

UMR 5669 CNRS – École Normale Supérieure de Lyon


Sur la coupe des vêtements.

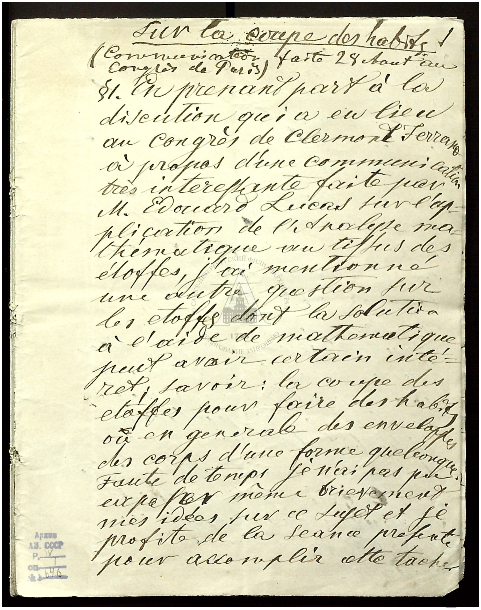
(Association française pour l'avancement des sciences. 7 session. Paris. Séance du 28 août 1878).

Après avoir indiqué que l'idée de cette étude lui est venue lors de la communication faite, il y a deux ans, au Congrès de Clermont-Ferrand, par M. Edouard Lucas, sur la géometrie du tissage des étoffes à fils rectilignes, M. Tchébichef pose les principes généraux pour déterminer les courbes suivant lesquelles on doit couper les differents morceaux d'une étoffe, pour en faire une gaine bien ajustée, servant à envelopper un corps de forme quelconque.

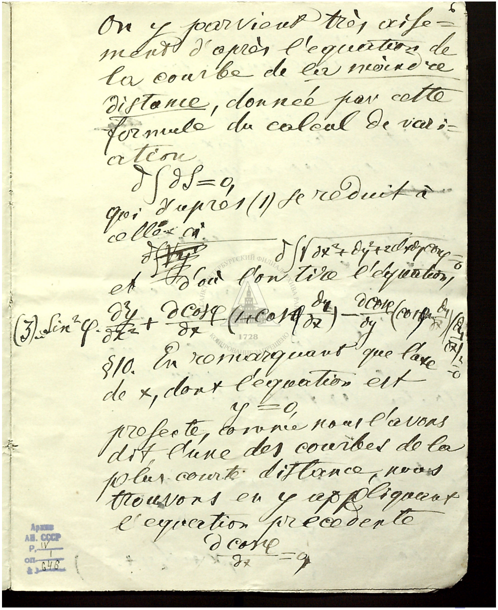
En prenant pour point de départ ce principe d'observation que dans la déformation d'un tissu on ne doit considérer d'abord, dans une première approximation, que l'altération des angles respectifs formés par les fils de chaîne et les fils de trame, sans tenir compte de l'allongement des fils, il donne les formules qui permettent de déterminer les contours imposés à deux, trois ou quatre morceaux d'étoffe pour recouvrir la surface d'une sphère, avec la meilleure approximation désirable. M. Tchébichef présente à la section une balle de caoutchouc recouverte d'une étoffe dont les deux morceaux ont été coupés suivant ses indications; il fait observer que le problème différerait essentiellement si l'on remplaçait l'étoffe par une peau. D'ailleurs les formules proposées par M. Tchébichef donnent aussi la méthode à suivre pour la juxtaposition des pièces par la couture.

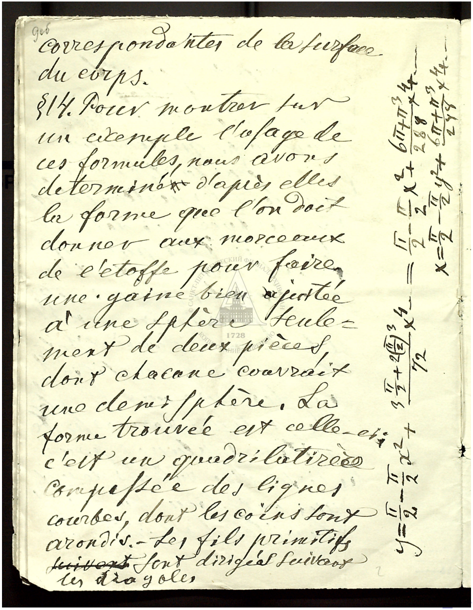
Conformément à la volonté de Tchebychef, l'étude «Sur la coupe des habits» trouvée dans ses papiers ne doit pas être imprimée, car le manuscrit ne porte pas l'inscription: «*imprimer*».

La coupe des habits


Sur la coupe des habits !
(Communication faite 28 Août au Congrès de Paris)
§1. En prenant part à la discution qui a eu lieu au Congrès de Clermont-Ferrand à propos d'une communication très interessante faite par M. Edouard Lucas sur l'application de l'Analyse mathématique au tissus des étoffes, j'ai mentionné une autre question sur les étoffes dont la solution à l'aide de mathématique peut avoir certain intéret, savoir : la coupe des étoffes pour faire des habits ou en generale des enveloppes des corps d'une forme quelconque. Faute de temps je n'ai pas pu exposer même brièvement mes idées sur ce sujet et je profite de la séance présente pour accomplir cette tache.

La coupe des habits



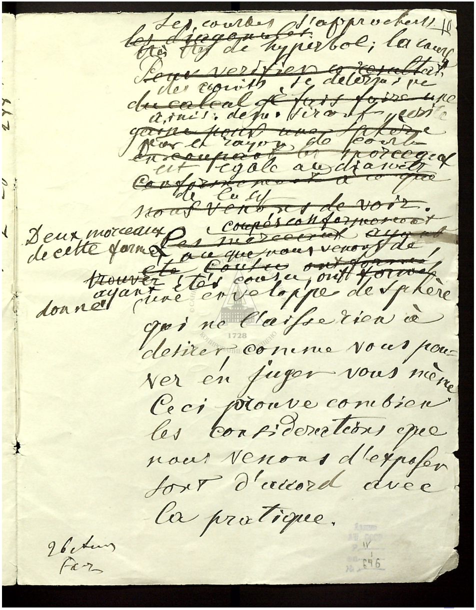


correspondantes de la surface
du corps.
§14. Pour montrer sur
un exemple l'usage de
ces formules, nous avons
déterminé d'après elles
la forme que l'on doit
donner aux morceaux
de l'étoffe pour faire
une gaine bien ajustée
à une sphère seule-
ment de deux pièces,
dont chacune couvrait
une demi sphère. La
forme trouvée est celle-ci
c'est un quadrilatère
composée des lignes
courbes, dont les coins sont
arrondis. — Les fils primitifs
~~suivant~~ sont dirigés suivant
les diagonales

$$y=\frac{\pi}{2}-\frac{\pi}{2}x^2+\frac{3\frac{\pi}{2}+2\left(\frac{\pi}{2}\right)^3}{72}x^4 = \frac{\pi}{2}-\frac{\pi}{2}x^2+\frac{6\pi+\pi^3}{288}x^4$$

$$x=\frac{\pi}{2}-\frac{\pi}{2}y^2+\frac{6\pi+\pi^3}{288}y^4$$

La coupe des habits

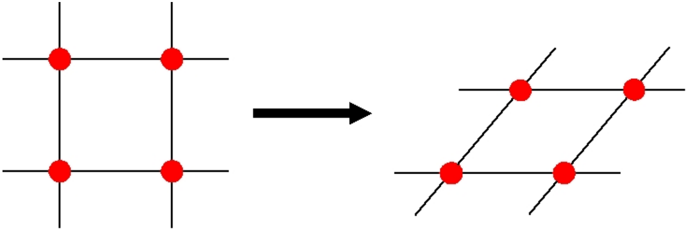


Les courbes s'approchent très très de hyperbol; la courbe

Deux morceaux de cette forme coupés conformement à ce que nous venons de trouver ayant été cousu, ont donné une enveloppe de sphère qui ne laisse rien à desirer comme vous pouvez en juger vous même. Ceci prouve combien les considerations que nous venons d'exposer sont d'accord avec la pratique.

Cloth

Small squares are deformed into small rhombuses.



Which surfaces can be "clothed" ?

Definition : *A surface is "clothable" if it can be parameterized by some* $\Phi : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ *such that the vectors* $\partial\Phi/\partial x$ *and* $\partial\Phi/\partial y$ *have norm 1(but maybe non orthogonal).*

$$g = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2$$

$a = c, b = 0$: conformal coordinates.

$a = 1, c = 1$: Chebyshev net.

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Clothable surfaces

Theorem : *ALL surfaces are LOCALLY clothable.*

We may need a huge number of pieces to cloth a given surface.

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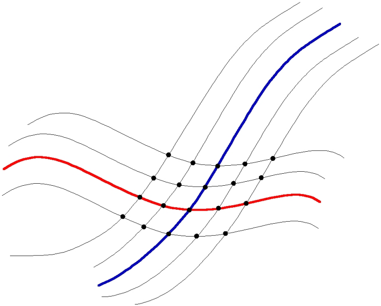
"Proof : "

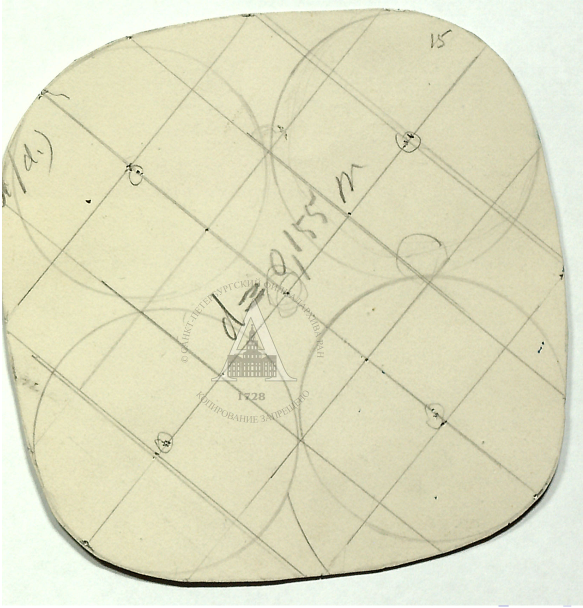


Hammock

A theorem by Chebyshev ?

Theorem : *On can cloth a sphere with TWO pieces, each covering a half sphere.*

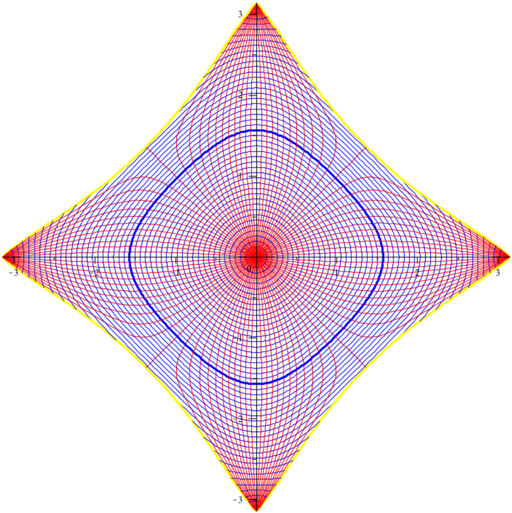
La coupe des habits



Better...

Theorem : *On can cloth a sphere with ONE piece.*

The template



Clothing the sphere

Theorem [Bakelman, Samelson] : *A half sphere can be clothed.*

Theorem [Burago, Ivanov, Malev] : *A simply connected surface such that* $\int K^+ < 2\pi$ *and* $\int K^- < 2\pi$ *can be clothed.*

Theorem [Bakelman, Samelson] : *A half sphere can be clothed.*

Theorem [Burago, Ivanov, Malev] : *A simply connected surface such that* $\int K^+ < 2\pi$ *and* $\int K^- < 2\pi$ *can be clothed.*

A few words about the proofs

$$\frac{\partial^2\Phi}{\partial x\partial y} = \frac{\partial}{\partial y}\frac{\partial\Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\Phi}{\partial y}$$

$\frac{\partial^2\Phi}{\partial x\partial y}$ is orthogonal to the surface.

Hence $\frac{\partial\Phi}{\partial x}$ is parallel along the y-curves.

The geodesic curvature of the x-curves is $\frac{\partial\omega(x,y)}{\partial x}$.
The geodesic curvature of the y-curves is $-\frac{\partial\omega(x,y)}{\partial y}$.

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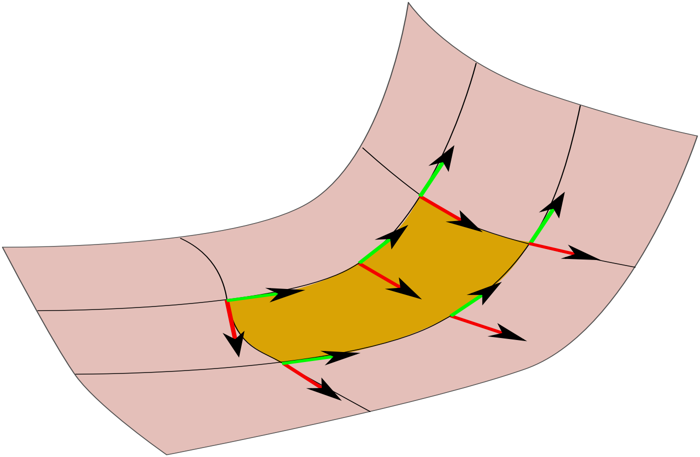
$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} K(u,v)\sin(\omega(x,y))\,dxdy = \omega(x_0,y_0)-\omega(x_0,y_1)+\omega(x_1,y_1)-\omega(x_1,y_0).$$

Gauss-Bonnet formula



Hazzidakis formulas

$$\frac{\partial^2 \omega}{\partial x \partial y} = -K(x,y) \sin \omega(x,y).$$

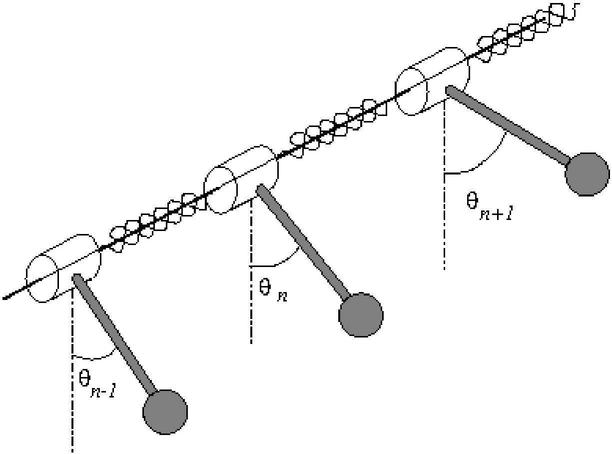
A partial differential equation

If one wants to cloth the sphere, the angle $\omega(x, y)$ has to satisfy the following PDE.

$$\frac{\partial^2 \omega}{\partial x \partial y} = -\sin \omega.$$

This is sine-Gordon PDE.

sine-Gordon pde



$$\frac{\partial^2\theta}{\partial t^2} = \frac{\partial^2\theta}{\partial x^2} + \sin\theta.$$

There is a unique solution of this PDE of the form $\omega(x,y) = U(xy)$ where U is a smooth function from $\mathbf{R}$ to $\mathbf{R}$ such that $U(0) = \pi/2$.

$$\frac{d^2U}{dx^2} + \frac{dU}{dx} + \sin U(x) = 0$$

$$U(x) = \frac{\pi}{2} - x + \frac{x^3}{18} + \cdots$$

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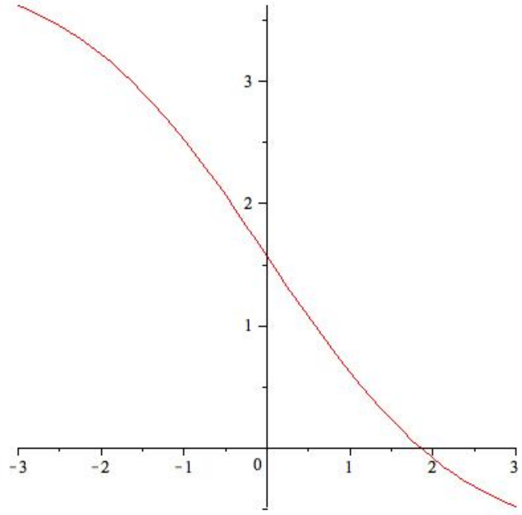
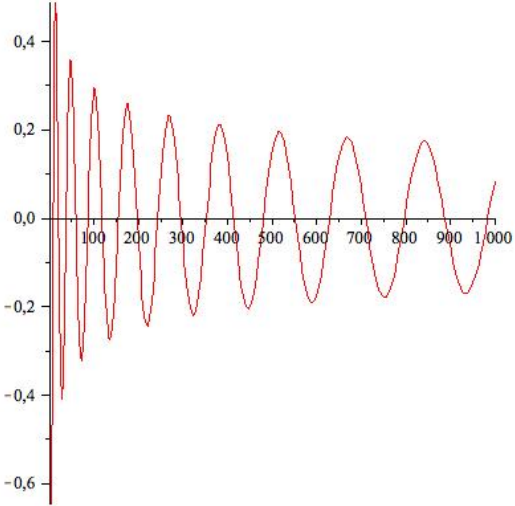
A series

$$
\begin{aligned}
U(x) = {} & \frac{\pi}{2} - x + \frac{1}{18}x^3 - \frac{7}{1800}x^5 \\
& + \frac{521}{1587600}x^7 - \frac{31139}{1028764800}x^9 \\
& + \frac{18279367}{6224027040000}x^{11} \\
& - \frac{11159392859}{37866980511360000}x^{13} \\
& + \frac{25289583956249}{834966920275488000000}x^{15} \\
& - \frac{4078693576473449}{1286962346451285504000000}x^{17} \\
& + \frac{15185544082366872679}{451584791670984473069568000000}x^{19} \\
& - \frac{21133178727426263957897}{585732038608541625363763200000000}x^{21} \\
& + \cdots
\end{aligned}
$$


3
2
1
-3
-2
-1
0
1
2
3


0,4
0,2
0,0
-0,2
-0,4
-0,6
100
200
300
400
500
600
700
800
900
1 000

The metric

$$g = dx^2 + 2\cos U(xy)dx\,dy + dy^2$$

is a non-degenerate Riemannian metric away from the hyperbolas $xy = \zeta$ where $U(\zeta) = k\pi$.

One shows that there exists a unique "isometry" Ψ from the plane $\mathbf{R}^2$ equipped with g to the round sphere....

$$\Psi : \mathbf{R}^2 \longrightarrow \mathbf{S}^2$$

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One establishes that there is an open set Ω in the plane (the template) such that the restriction of Ψ to Ω is a clothing of the sphere (minus four geodesic arcs).

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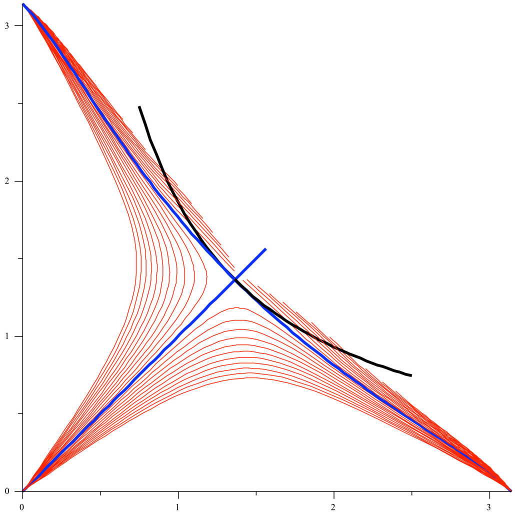
3
2
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0
0
1
2
3

A differential equation

$$x\frac{d^2U}{dx} + \frac{dU}{dx} + \sin U(x) = 0$$

$$W = \exp(-iU)$$

$$\frac{d^2W}{dx^2} = \frac{1}{W}\left(\frac{dW}{dx}\right)^2 - \frac{1}{x}\frac{dW}{dx} + \frac{1}{2}\frac{(1-W^2)}{x}$$

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$$\frac{d^2W}{dx^2} = R(x, W, \frac{dW}{dx})$$

Painlevé property :
"All movable singularities are poles"
Six Painlevé equations $PI, PII, ..., PVI$.
Painlevé III :

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Complex Hamiltonian differential equations.

"Conjecture" : Complete integrability for evolution pde is related to Painlevé...

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Complex clothing...

Theorem : *There is a meromorphic clothing* $\mathbf{C}^2 \to \{x^2 + y^2 + z^2 = 1\} \subset \mathbf{C}^3$.

The "complexified sphere" is $\mathbf{P}^1 \times \mathbf{P}^1 \setminus \Delta$ with the metric $\frac{dudv}{(u-v)^2}$.

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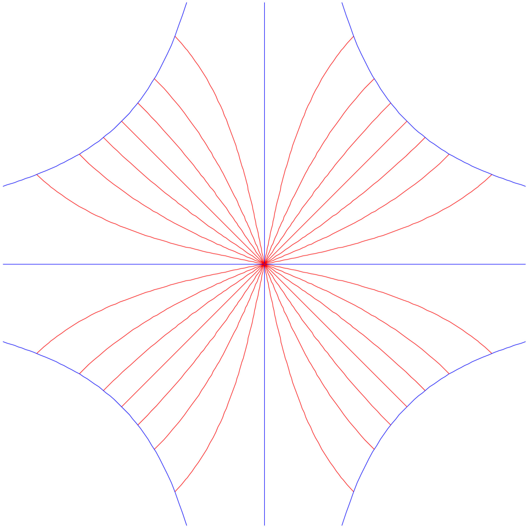
A remark

Changing signs :

$$\frac{\partial^2 \omega}{\partial x \partial y} = + \sin \omega,$$

We get a clothing of (a part) of the hyperbolic plane.

Clothing part of the hyperbolic plane.



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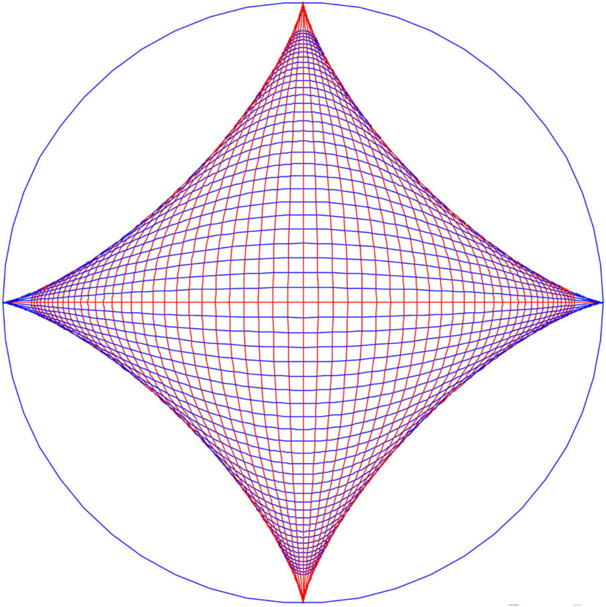
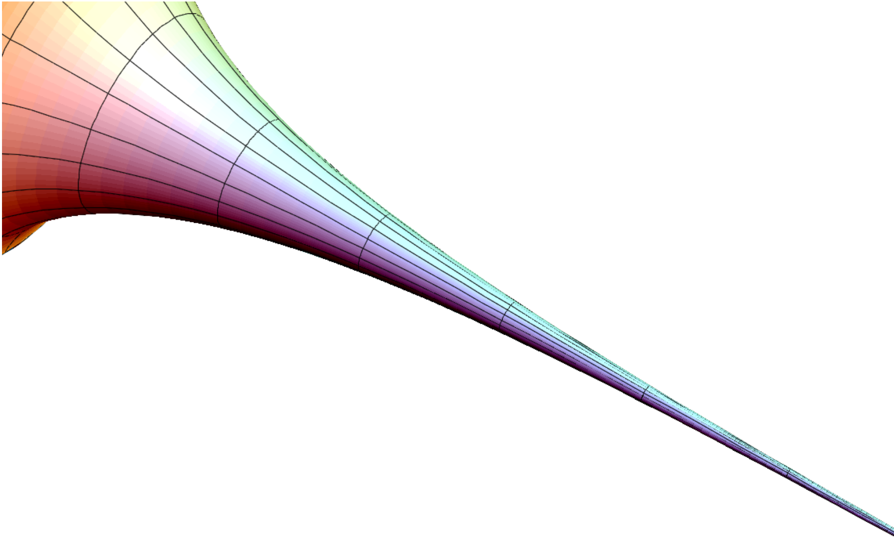


Embedding the hyperbolic plane in 3-space



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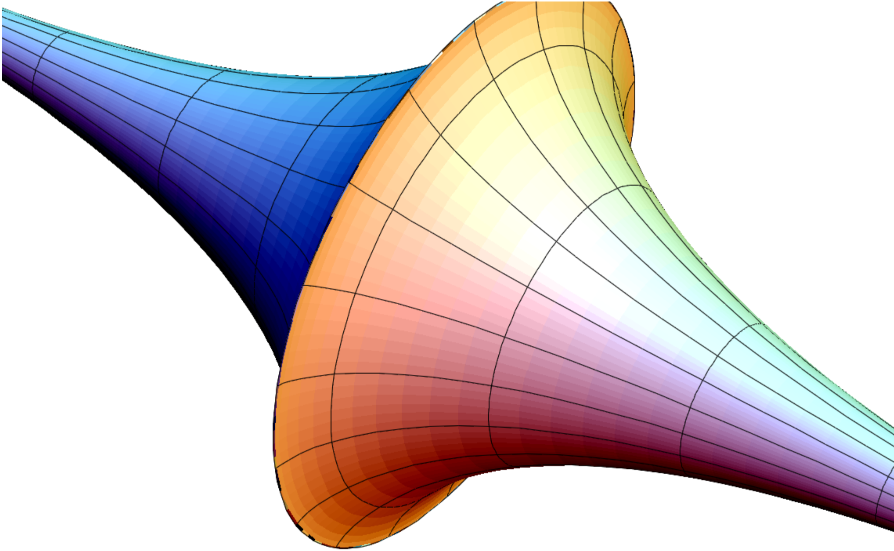

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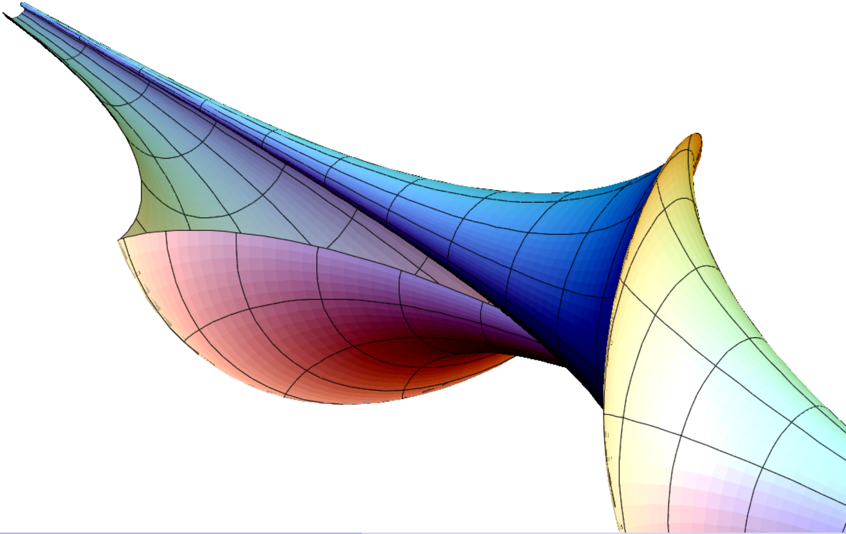

Theorem :

- *Asymptotic lines on a surface of curvature* -1 *define a Chebyshev net.*
- *The Gauss map of a surface of curvature* -1 *defines a clothing of (part of) the sphere.*

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Hilbert's proposition

- $\overrightarrow{N}(p)$ unit vector orthogonal to S at p.
- Gauss map : $\overrightarrow{N} : S \to \mathbf{S}^2$.
- Shape operator on the tangent plane $A(\overrightarrow{v}) = D\overrightarrow{N}(\overrightarrow{v})$
- Second fundamental form on the tangent plane $II_p(\overrightarrow{v}) = A(\overrightarrow{v}) \cdot \overrightarrow{v}$
- Eigenvalues are the principal curvatures.
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- Shape operator on the tangent plane $A(\overrightarrow{v}) = D\overrightarrow{N}(\overrightarrow{v})$
- Second fundamental form on the tangent plane $II_p(\overrightarrow{v}) = A(\overrightarrow{v}) \cdot \overrightarrow{v}$
- Eigenvalues are the principal curvatures.
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$$A = \begin{pmatrix} \alpha^2 & 0 \\ 0 & -\alpha^{-2} \end{pmatrix}.$$

- Second fundamental form is $\alpha^2 x^2 - \alpha^{-2} y^2$.
- Isotropic vectors are $f_1 = (\alpha^{-1}, -\alpha)$ and $f_2 = (\alpha^{-1}, \alpha)$.
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- Local parameter $(x, y) \mapsto F(x, y) \in \mathbf{R}^3$ so that asymptotic curves are $x = const$ and $y = const$.

$$\frac{\partial F}{\partial x} = -\overrightarrow{N} \wedge \frac{\partial \overrightarrow{N}}{\partial x} \quad ; \quad \frac{\partial F}{\partial y} = \overrightarrow{N} \wedge \frac{\partial \overrightarrow{N}}{\partial y}.$$

- Derivative w.r.t. y + derivative w.r.t. x :

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial \overrightarrow{N}}{\partial x} \wedge \frac{\partial \overrightarrow{N}}{\partial y}.$$

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Theorem : *There is no smooth embedding of the hyperbolic plane in Euclidean space* $\mathbf{R}^3$.

Hint : (Hazzidakis formula)

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} K(u,v)\sin(\omega(x,y))\,dxdy = \omega(x_0,y_0)-\omega(x_0,y_1)+\omega(x_1,y_1)-\omega(x_1,y_0).$$

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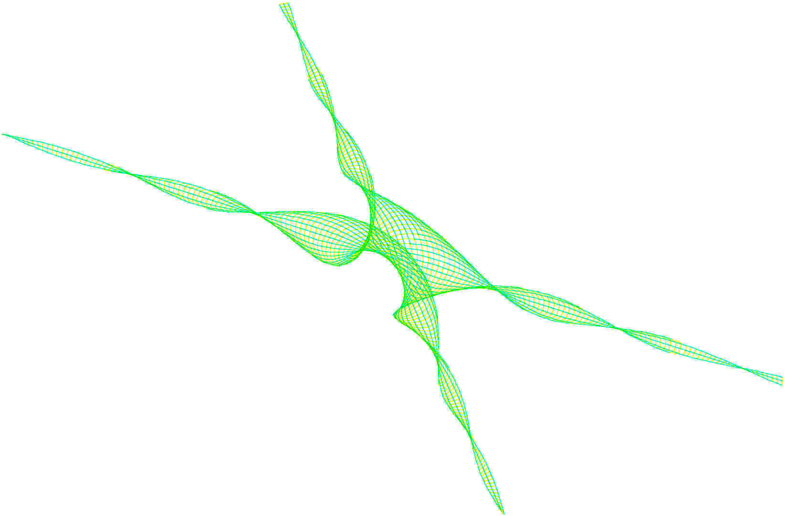
Amsler surface



Amsler surface

Amsler surface

Theorem : *This surface extends to a meromorphic map* $\mathbf{C}^2 \to \mathbf{C}^3$

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Clothing an ellipsoid

Back to clothable surfaces

Theorem : *Let S be a compact Riemannian surface (or more generally with bounded geometry). Let γ_1 and γ_2 be two geodesics intersecting at $\gamma_1(0) = \gamma_2(0)$. Then there is a unique globally defined "clothing" $\Phi : \mathbf{R}^2 \to S$ such that $\Phi(u,0) = \gamma_1(u)$ and $\Phi(0,v) = \gamma_2(v)$.*

Be careful, it may be the case that Φ is not onto.

Corollary : *A convex surface whose curvature is almost constant can be clothed with only one piece of cloth.*

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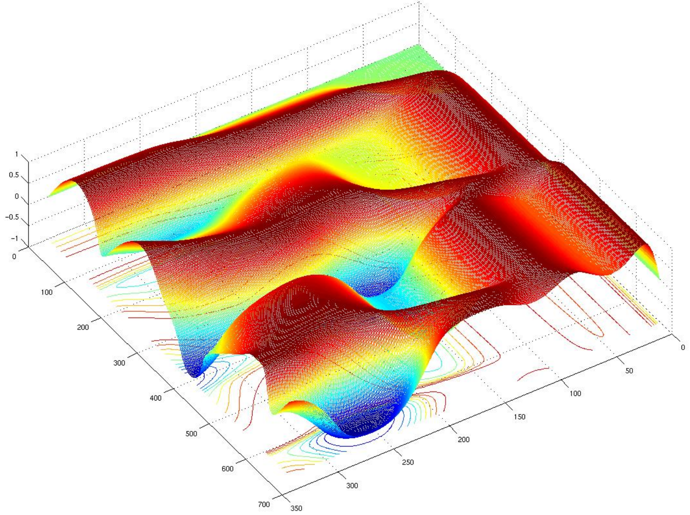
$$\frac{\partial^2 u}{\partial x \partial y} = \Gamma^1_{11} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \Gamma^1_{12} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) + \Gamma^1_{22} \frac{\partial v}{\partial x} \frac{\partial v}{\partial y}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \Gamma^2_{11} \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \Gamma^2_{12} \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) + \Gamma^2_{22} \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}$$

Ellipsoid



1
0.5
0
-0.5
-1
0
100
200
300
400
500
600
700
350
300
250
200
150
100
50
0

Ellipsoids

$$\frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = 1 \qquad (E(a, b, c))$$

$$ds^2 = \frac{1}{4}(u - v)\left(\frac{u}{(a-u)(b-u)(c-u)}du^2 - \frac{v}{(a-v)(b-v)(c-v)}dv^2\right)$$

Question : *Does the clothing*

$$\mathbf{R}^2 \to E(a, b, c)$$

extend to a holomorphic clothing

$$\mathbf{C}^2 \to \mathbf{P}^1 \times \mathbf{P}^1?$$

Is this given by some Painlevé III equation ?

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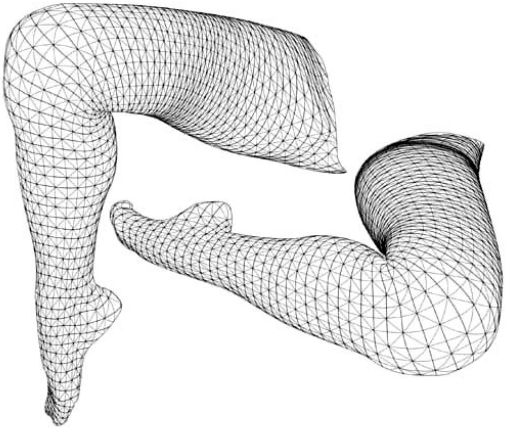

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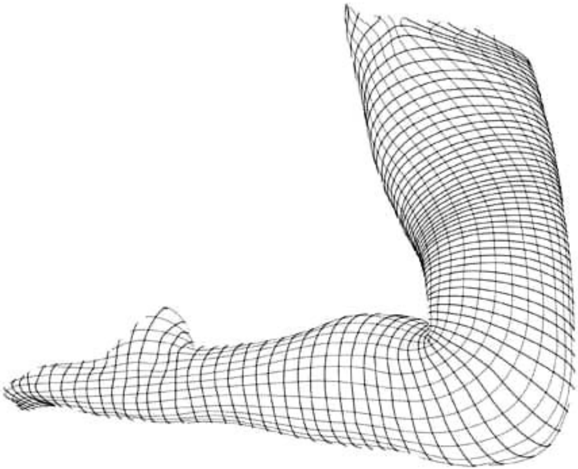

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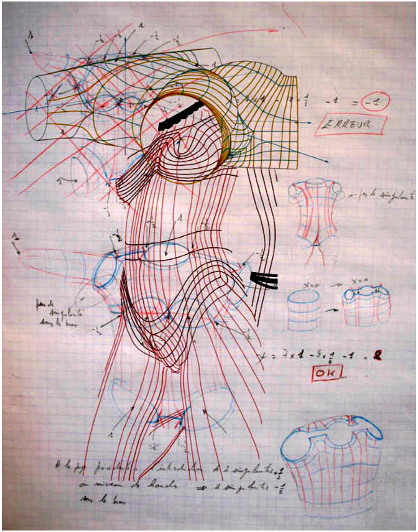

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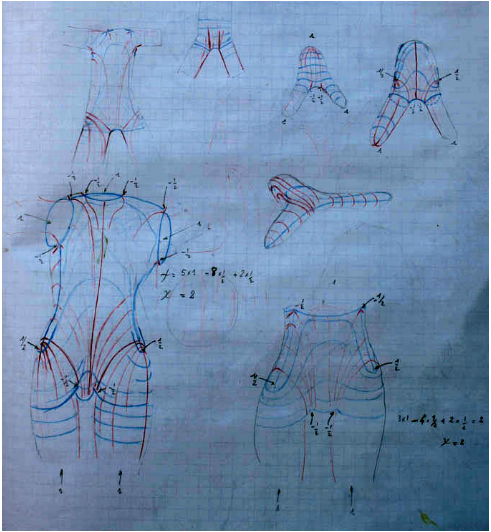

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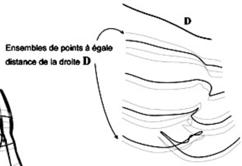
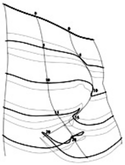
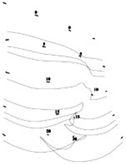
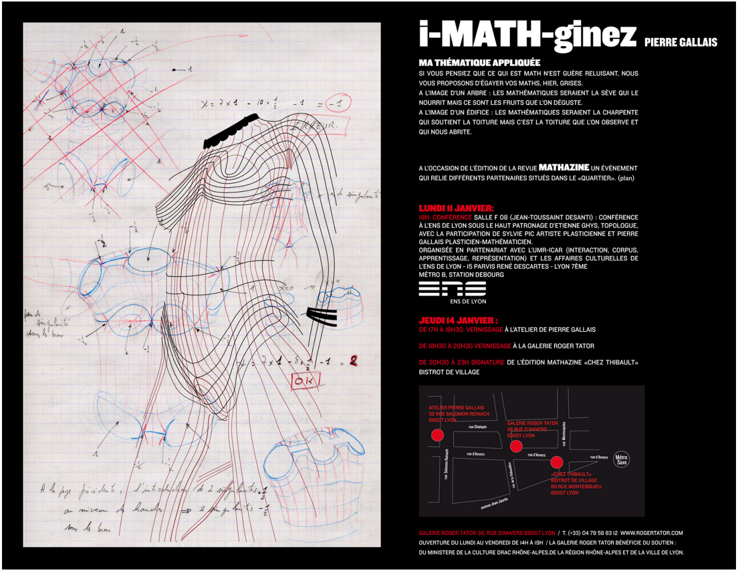






I-Math-ginez



i-MATH-ginez PIERRE GALLAIS
MA THÉMATIQUE APPLIQUÉE
SI VOUS PENSIEZ QUE CE QUI EST MATH N'EST GUÈRE RELUISANT, NOUS VOUS PROPOSONS D'ÉGAYER VOS MATHS, HIER, GRISES.
A L'IMAGE D'UN ARBRE : LES MATHÉMATIQUES SERAIENT LA SÈVE QUI LE NOURRIT MAIS CE SONT LES FRUITS QUE L'ON DÉGUSTE.
A L'IMAGE D'UN ÉDIFICE : LES MATHÉMATIQUES SERAIENT LA CHARPENTE QUI SOUTIENT LA TOITURE MAIS C'EST LA TOITURE QUE L'ON OBSERVE ET QUI NOUS ABRITE.
A L'OCCASION DE L'ÉDITION DE LA REVUE MATHAZINE UN ÉVÈNEMENT QUI RELIE DIFFÉRENTS PARTENAIRES SITUÉS DANS LE «QUARTIER». (plan)
LUNDI 11 JANVIER:
18H: CONFÉRENCE SALLE F 08 (JEAN-TOUSSAINT DESANTI) : CONFÉRENCE À L'ENS DE LYON SOUS LE HAUT PATRONAGE D'ETIENNE GHYS, TOPOLOGUE, AVEC LA PARTICIPATION DE SYLVIE PIC ARTISTE PLASTICIENNE ET PIERRE GALLAIS PLASTICIEN-MATHÉMATICIEN.
ORGANISÉE EN PARTENARIAT AVEC L'UMR-ICAR (INTERACTION, CORPUS, APPRENTISSAGE, REPRÉSENTATION) ET LES AFFAIRES CULTURELLES DE L'ENS DE LYON - 15 PARVIS RENÉ DESCARTES - LYON 7ÈME
MÉTRO B, STATION DEBOURG
ENS DE LYON
JEUDI 14 JANVIER :
DE 17H À 18H30: VERNISSAGE À L'ATELIER DE PIERRE GALLAIS
DE 18H30 À 20H30 VERNISSAGE À LA GALERIE ROGER TATOR
DE 20H30 À 23H SIGNATURE DE L'ÉDITION MATHAZINE «CHEZ THIBAULT» BISTROT DE VILLAGE
ATELIER PIERRE GALLAIS
52 RUE SALOMON REINACH
69007 LYON
GALERIE ROGER TATOR
36 RUE D'ANVERS
69007 LYON
«CHEZ THIBAULT»
BISTROT DE VILLAGE
80 RUE MONTESQUIEU
69007 LYON
rue Chalopin
rue d'Anvers
avenue Jean Jaurès
Métro Saxe
GALERIE ROGER TATOR 36, RUE D'ANVERS 69007 LYON / T. (+33) 04 78 58 83 12 WWW.ROGERTATOR.COM
OUVERTURE DU LUNDI AU VENDREDI DE 14H À 19H / LA GALERIE ROGER TATOR BÉNÉFICIE DU SOUTIEN :
DU MINISTERE DE LA CULTURE DRAC RHÔNE-ALPES,DE LA RÉGION RHÔNE-ALPES ET DE LA VILLE DE LYON.