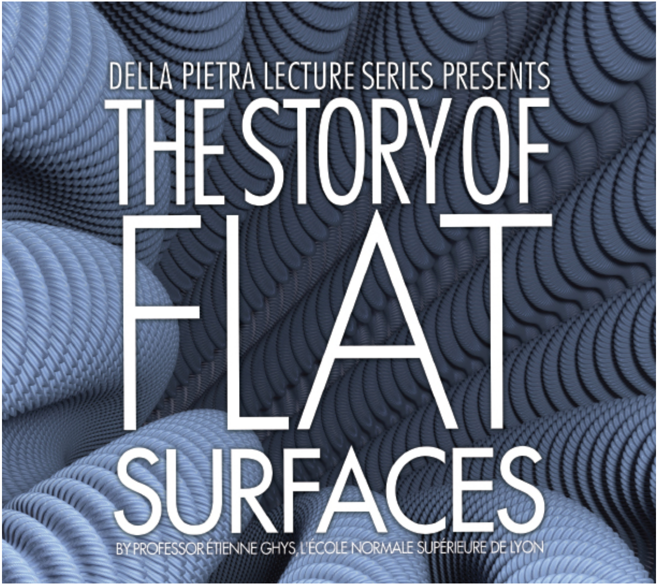

DELLA PIETRA LECTURE SERIES PRESENTS
THE STORY OF
FLAT
SURFACES
BY PROFESSOR ÉTIENNE GHYS, L'ÉCOLE NORMALE SUPÉRIEURE DE LYON

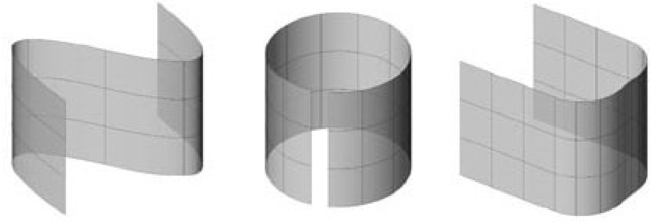
Which surfaces can be covered with paper ?

Leonhard Euler (1707-1783)

Which surfaces can be covered with paper,
allowing bending but not stretching ?

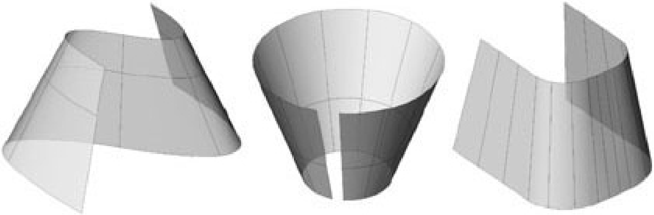
Leonhard Euler (1707-1783)

Which surfaces can be clothed with paper?



Cylinders

Which surfaces can be clothed with paper ?



Cones

Paper Sculpture Fashion by Zaharova and Plotnikov

Paper Sculpture Fashion by Zaharova and Plotnikov

DE

SOLIDIS QVORVM

SVPERFICIEM IN PLANVM EXPLICARE LICET.

Auctore

L. EVLERO.

1.

Notiſſima eſt proprietas cylindri et coni, qua eorum ſuperficiem in planum explicare licet atque adeo haec proprietas ad omnia corpora cylindrica et conica extenditur, quorum baſes figuram habeant quamcunque; contra vero ſphaera hac proprietate deſtituitur, quum eius ſuperficies nullo modo in planum explicari neque ſuperficie plana obduci queat; ex quo naſcitur quaeſtio aeque curioſa ac notatu digna, vtrum praeter conos et cylindros alia quoque corporum genera exiſtant, quorum ſuperficiem itidem in planum explicare liceat nec ne? quam ob rem in hac diſſertatione ſequens conſiderare conſtitui Problema:

Inuenire aequationem generalem pro omnibus ſolidis, quorum ſuperficiem in planum explicare licet, cuius ſolutionem variis modis ſum agreſſurus.

Euler, Saint Peterburg, 1770

Definition : *A surface is developable if it can be covered with paper allowing bending but not stretching.*

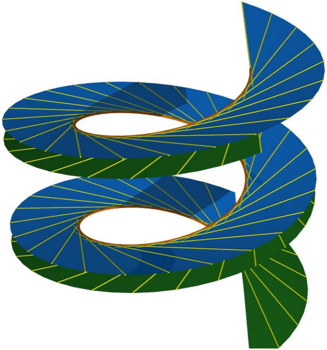
Theorem (Euler 1770) : *A developable surface is necessarily* **ruled**. *Through every point there is a straight line that lies on the surface.*

Definition : *A surface is developable if it can be covered with paper allowing bending but not stretching.*

Theorem (Euler 1770) : *A developable surface is necessarily* **ruled**. *Through every point there is a straight line that lies on the surface.*

Most ruled surfaces are not developable.

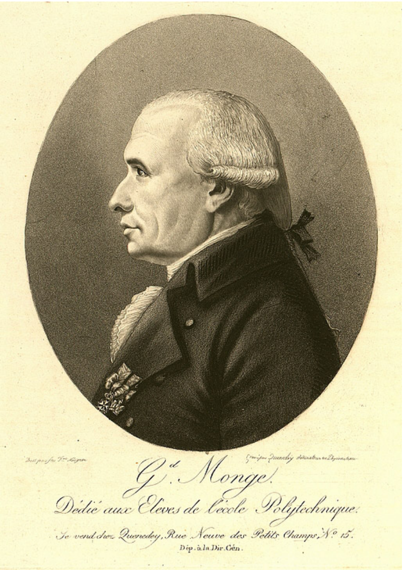
Theorem (Euler 1770) : *A ruled surface is developable if and only if it is either a cone or a cylinder or is formed by the union of the tangents to some space curve.*



Antoine-Pevsner-Developable-Surface-1938

Antoine-Pevsner-Developable-Surface-1938

Gaspard Monge 1746-1818



Mémoire

Sur les Propriétés de plusieurs genres de Surfaces courbes, particulièrement sur celles des Surfaces développables, avec une application à la théorie générale des Ombres et des pénombres.

Dans un Mémoire que j'eus l'honneur en 1771 de présenter à l'académie je fis voir qu'il n'y avoit aucune courbe, plane ou à double courbure, qui n'eut une infinité de dévelopées ; que toutes ces dévelopées étoient à double courbure, à l'exception d'une seule pour chaque courbe plane : que l'assemblage, ou plutôt le lieu géométrique de toutes les dévelopées d'une même courbe formoit une surface courbe qui jouissoit de la propriété de pouvoir être étendue sur un plan de manière à le toucher dans tous ses points ; Par exemple le lieu des dévelopées d'une courbe sphérique quelconque est toujours la surface d'un cone, dont le sommet est au centre de la sphère, et dont la base dépend de la nature de la courbe. Je démontrai qu'en supposant construit le lieu des dévelopées d'une courbe on obtenoit une de ses dévelopées quelconque, en menant

Theorem (Monge) : *A developable surface is an envelope of a family of planes.*

Developable surface

Developable surface

http ://www.omkrets.se

Henri Lebesgue 1875-1941

Lebesgue's handkerchief : counterexample to Euler's theorem ?

Theorem (Lebesgue 1899) : Some surfaces are developable and not ruled !

NOT smooth. Continuous surfaces with no tangent planes.

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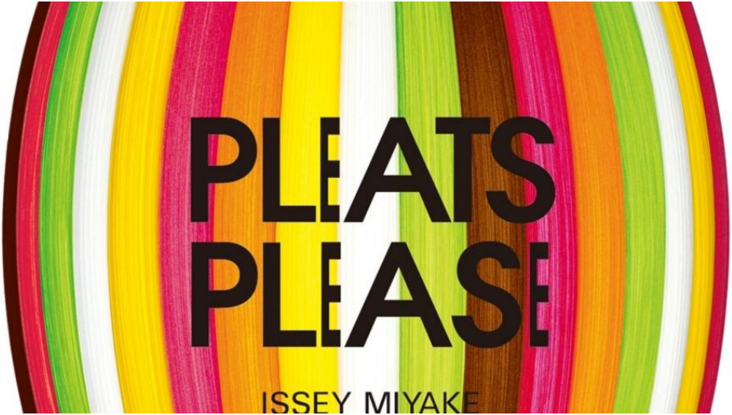
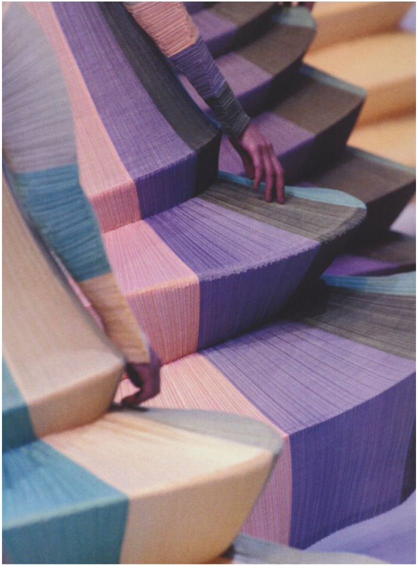
Issey Miyake (1938-)


PLEATS
PLEASE
ISSEY MIYAKE

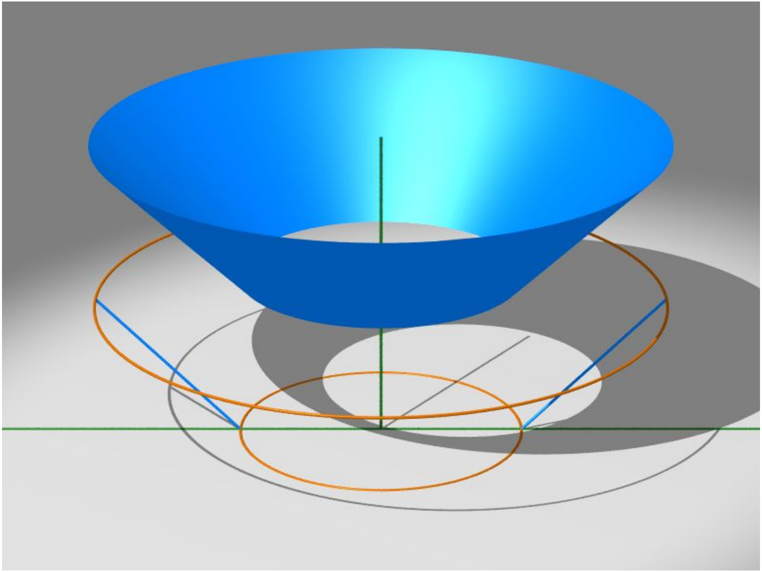
Issey Miyake *Minaret dress* : Developable surface ? Ruled ?

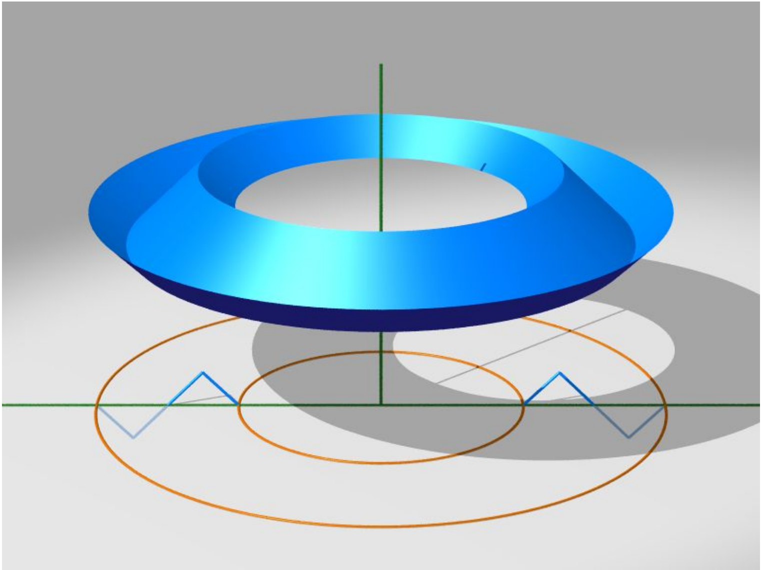


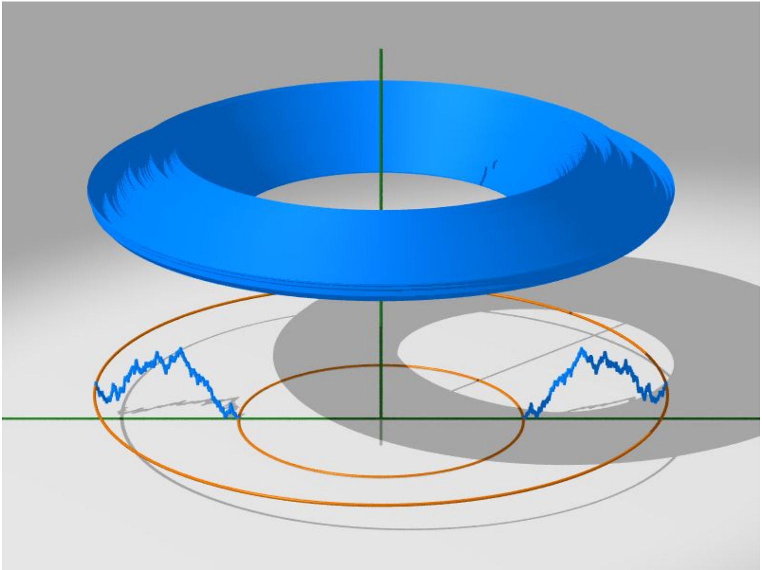
Issey Miyake *Minaret dress* : Developable surface ? Ruled ?

Issey Miyake *Minaret dress* : Developable surface ? Ruled ?

A developable surface, Issey Miyake Milan 1995







John Nash (1928-)

Some surfaces are developable, smooth AND not ruled !

Theorem (Nash 1954) : *There exists a surface which*

- *is smooth, i.e. has a tangent plane everywhere,*
- *is developable : can be clothed with paper,*
- *is not ruled : it contains no segment.*

They are smooth, with no tangent planes but they do not have a second derivative. Euler theorem requires two derivatives.

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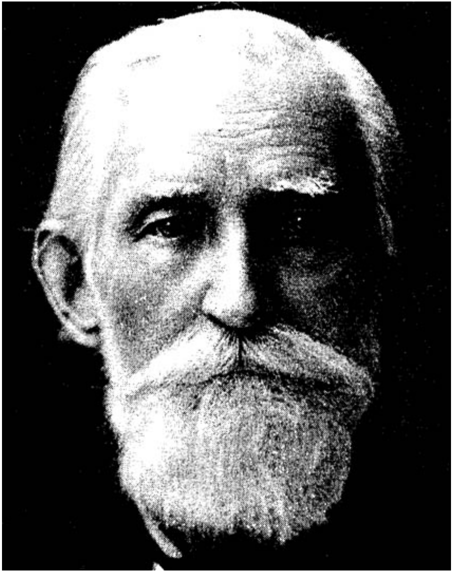
Flat, smooth and not ruled

Chebyshev
Cloth, not paper !



Pafnuti Chebyshev (1821-1894)

Sur la coupe des vêtements.

(Association française pour l'avancement des sciences. 7 session. Paris. Séance du 28 août 1878).

Après avoir indiqué que l'idée de cette étude lui est venue lors de la communication faite, il y a deux ans, au Congrès de Clermont-Ferrand, par M. Edouard Lucas, sur la géometrie du tissage des étoffes à fils rectilignes, M. Tchébichef pose les principes généraux pour déterminer les courbes suivant lesquelles on doit couper les differents morceaux d'une étoffe, pour en faire une gaine bien ajustée, servant à envelopper un corps de forme quelconque.

En prenant pour point de départ ce principe d'observation que dans la déformation d'un tissu on ne doit considérer d'abord, dans une première approximation, que l'altération des angles respectifs formés par les fils de chaîne et les fils de trame, sans tenir compte de l'allongement des fils, il donne les formules qui permettent de déterminer les contours imposés à deux, trois ou quatre morceaux d'étoffe pour recouvrir la surface d'une sphère, avec la meilleure approximation désirable. M. Tchébichef présente à la section une balle de caoutchouc recouverte d'une étoffe dont les deux morceaux ont été coupés suivant ses indications; il fait observer que le problème différerait essentiellement si l'on remplaçait l'étoffe par une peau. D'ailleurs les formules proposées par M. Tchébichef donnent aussi la méthode à suivre pour la juxtaposition des pièces par la couture.

Conformément à la volonté de Tchebychef, l'étude «Sur la coupe des habits» trouvée dans ses papiers ne doit pas être imprimée, car le manuscrit ne porte pas l'inscription: «*imprimer*».

Sur la coupe des habits. 1

(Communication faite 28 Aout au Congrès de Paris)

§1. En prenant part à la discution qui a eu lieu au Congrès de Clermont-Ferrand à propos d'une communication très interessante faite par M. Edouard Lucas sur l'application de l'Analyse mathématique au tissus des étoffes, j'ai mentionné une autre question sur les étoffes dont la solution à l'aide de mathematique peut avoir certain intérêt, savoir: la coupe des étoffes pour faire des habits où en generale des enveloppes des corps d'une forme quelconque. Faute de temps je n'ai pas pu exposer même brievement mes idées sur ce sujet et je profite de la seance présente pour accomplir cette tache.

On y parvient très aisément d'après l'equation de la courbe de la moindre distance, donnée par cette formule du calcul de variation

$$\delta\int ds = 0,$$

qui d'après (1) se reduit à celle-ci

$$\delta\int\sqrt{dx^2+dy^2+2dx\,dy\cos\varphi} = 0$$

et d'où l'on tire l'équation,

$$(3)\ \sin^2\varphi\cdot\frac{d^2y}{dx^2}+\frac{d\cos\varphi}{dx}\left(1+\cos\varphi\frac{dy}{dx}\right)-\frac{d\cos\varphi}{dy}\left(\cos\varphi+\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)^2 = 0$$

§10. En remarquant que l'axe de x, dont l'équation est

$$y = 0,$$

profecte, comme nous l'avons dit, l'une des courbes de la plus courte distance, nous trouvons en y appliquant l'equation precedente

$$\frac{d\cos\varphi}{dx} = 0,$$

correspondantes de la surface du corps.

§14. Pour montrer sur un exemple l'usage de ces formules, nous avons déterminé d'après elles la forme que l'on doit donner aux morceaux de l'étoffe pour faire une gaine bien ajustée à une sphère seulement de deux pièces, dont chacune couvrirait une demi sphère. La forme trouvée est celle-ci : c'est un quadrilatère composée des lignes courbes, dont les coins sont arondis. - Les fils primitifs ~~suivent~~ sont dirigés suivant les diagoles

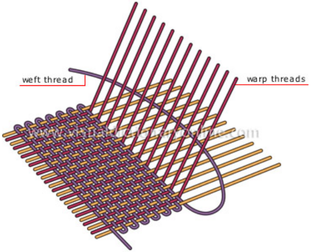
$$y=\frac{\pi}{2}-\frac{\pi}{2}x^2+\frac{3\frac{\pi}{2}+2\left(\frac{\pi}{2}\right)^3}{72}x^4 \ldots =\frac{\pi}{2}-\frac{\pi}{2}x^2+\frac{6\pi+\pi^3}{288}x^4 \ldots$$

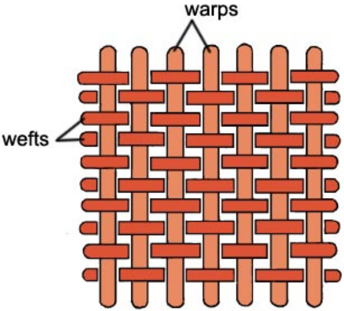
$$x=\frac{\pi}{2}-\frac{\pi}{2}y^2+\frac{6\pi+\pi^3}{288}y^4 \ldots$$

Les courbes s'approchent
~~les diagonales~~ très très de hyperbol; la courbe
~~Pour vérifier ce résultat~~
des e[illegible] je détermine
~~du calcul je fais faire une~~
a.vais. de p. Virant y [illegible]
~~garnie pour une sphère~~
par le rayon de courbe
~~en coupant les morceaux~~
est égale au diamètre
~~Conformément à ce que~~
de la sp
~~nous venons de voir~~.

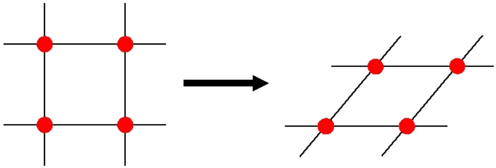
Deux morceaux de cette forme
~~coupés conformément~~
~~Les morceaux ayant~~
à ce que nous venons de
~~été coulés ont formé~~
~~trouver~~ ayant donné étés coupés, ont ~~formé~~
une enveloppe de sphère
qui ne laisse rien à
désirer, comme vous pou-
vez en juger vous même.
Ceci prouve combien
les considérations que
nous venons d'exposer
sont d'accord avec
la pratique.

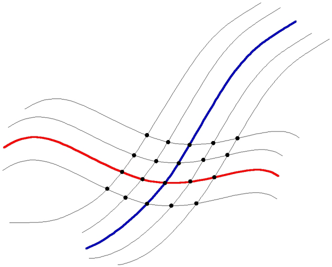
26 Juin
Far

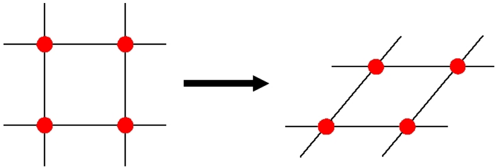


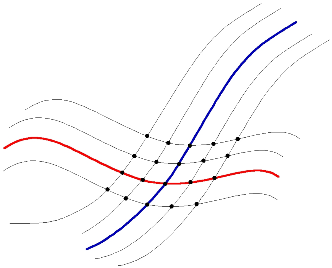


fab.cba.mit.edu

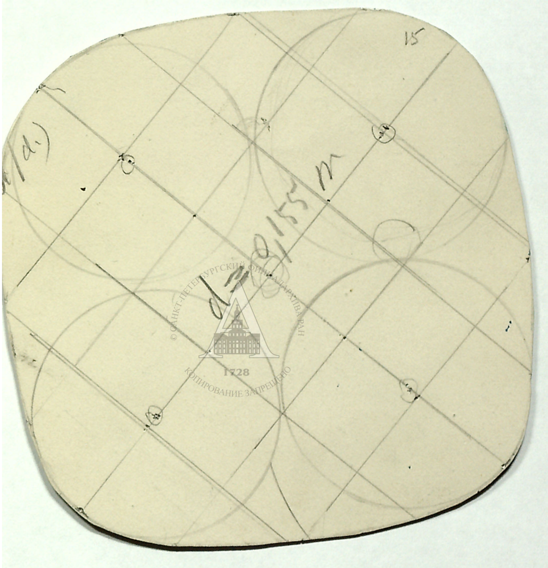






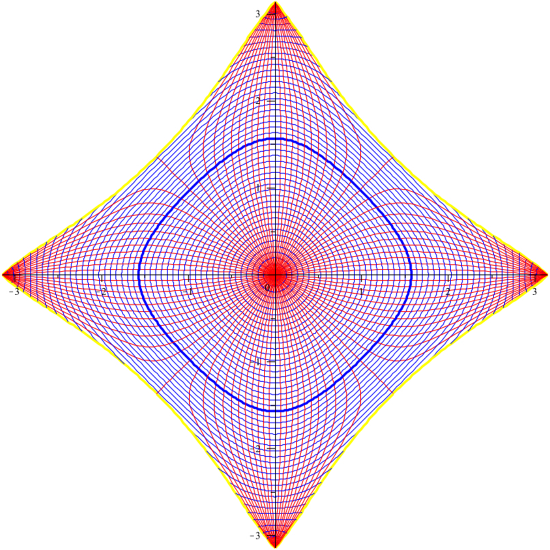


Theorem (Chebyshev 1878) *It is possible to clothe half the sphere. There exists a template that can be deposited on an hemisphere in such a way that the infinitesimal warp-weft squares are mapped to rhombuses.*

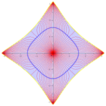


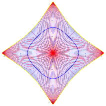
A template for the hemisphere

Theorem (G) *It is possible to clothe* **the full** *sphere. There exists a template that can be deposited on a sphere in such a way that the infinitesimal warp-weft squares are mapped to rhombuses.*



A template for the sphere

There is a map Φ from  to the sphere  such that :

- Φ preserves length along warp and weft, ie $\frac{\partial \Phi}{\partial x}$ and $\frac{\partial \Phi}{\partial y}$ have length 1.
- The image of Φ covers the sphere exactly once (away from the seams, on the boundary of ).

Leys

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

3 D printing of Chebyshev

3D printed dress by Michael Schmidt and Francis Bitonti

Dita Von Teese

3D printed dress by Michael Schmidt and Francis Bitonti

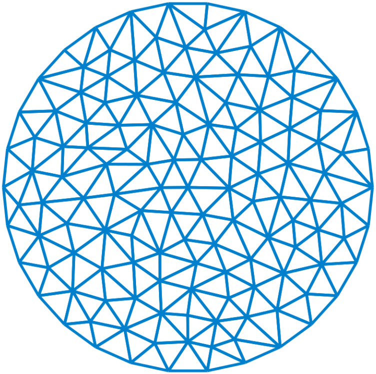
Dita Von Teese

3D printed dress by Michael Schmidt and Francis Bitonti

Dita Von Teese

Nervous.com

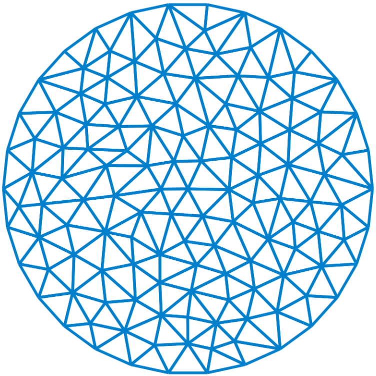
A triangulated disk is flexible.



$$\sharp\text{Degrees of freedom} = 3\sharp\text{Vertices} - \sharp\text{Edges} - 6$$

$$= \sharp\text{Exterior Edges} - 3$$

A triangulated disk is flexible.



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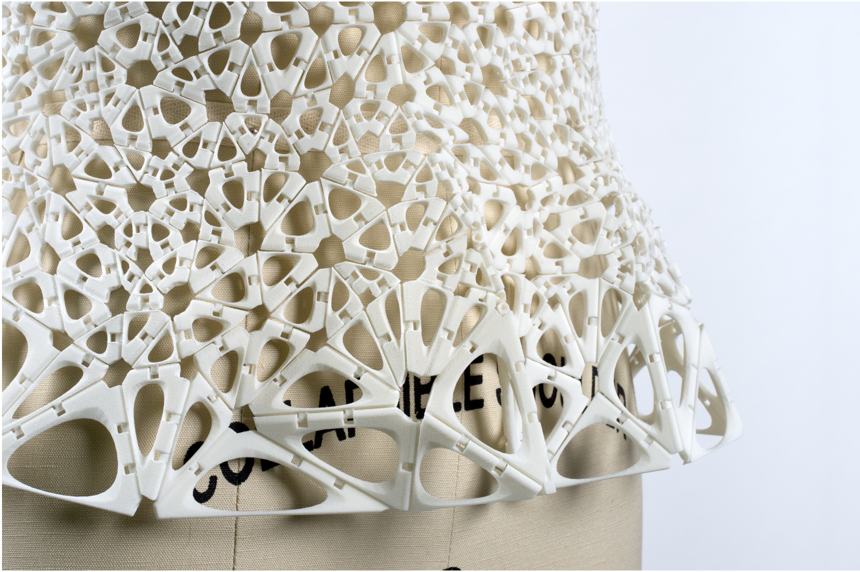
6
6



Zeeman and catastrophe theory

Sir Erik Christopher Zeeman (1925-)

Costume, Vol. 28, 97-102 (1994)

Mathematics Applied to Dressmaking

A Lecture Given at Gresham College, 9 March 1993

By PROFESSOR SIR CHRISTOPHER ZEEMAN, FRS

Gresham Professor of Geometry

DRESSMAKING can raise interesting questions in both geometry and topology. My own involvement began in Bangkok, where I once bought a dress-length of some rather beautiful Thai silk. Unfortunately when I got home all the dressmakers claimed it wasn't long enough to make a dress. It became clear that I had to either abandon the project or make the thing myself.

Now, I had never made a dress before, nor any garment for that matter, and so I thought it would be amusing to try designing it from scratch. Fools rush in where angels fear to tread. In my innocence I chose a simple sleeveless summer dress with a princess line; or in layman's language it just consisted of two panels, one at the front and one at the back, sewn together at the sides. What I had not realized was that, the simpler the dress, the more accurately it has to fit. As a precaution I first tried a mock-up made out of an old sheet. And a good thing, too, because the result was hopeless: when she tried it on it hardly fitted anywhere. I slowly began to realize that I did not yet understand the basic mathematical problem of how to fit a flexible flat surface round a curved surface. So back to the drawing board to do a little differential geometry.

Cutting 'on the cross' or 'on the bias' merely means arranging things so that the fibres of the warp and weft go diagonally relative to the body instead of going horizontally and vertically. Therefore, if the dress is pulled tight horizontally and vertically, then the material will be stretched diagonally relative to the fibres. Now, it is a characteristic property of woven material that it cannot be stretched *along* the fibres but it can be stretched *diagonal* to them. And if it is stretched along both diagonals at the same time then it will form a surface of negative curvature like a quadric, such as $xy = z$ (see Fig. 1). In a quadric the fibres are in fact straight lines (as in a plane), but they are no longer parallel to one another, and consequently they must have been pulled slightly apart round the perimeter (or pushed slightly closer together in the middle, or both). Incidentally, it is a classical theorem of projective geometry that any non-planar surface containing two intersecting families of lines is a quadric.

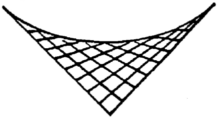


FIG. 1.

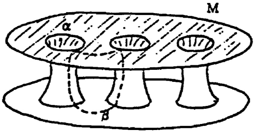


FIG. 7.

movement the surface occupies exactly the same position that it did at the beginning, except that individual points on the surface have been moved around. The surprising feature is the reversal of the sides. A similar isotopy is illustrated in Fig. 6, but that is easier to understand because there the surface is only of genus 1. In Fig. 7 if we identify the top of M with the dress, the bottom of M with the lining, and the three holes with the neck and armholes, then the theorem seems to imply that it is topologically possible to turn the dress inside out, after all. It was only after some thought that I realized the isotopy would also move the curve α into the curve β, because α can be spanned by a disk on only one side of the surface, while β can be spanned on the other. Therefore if the isotopy were applied to the dress, then the identity of the neck and arm holes would be lost. Hence the correct topological obstruction is: there is no side-reversing isotopy of the *pair* (M, α).

Note *by Lady Zeeman*

The length of cloth brought back from Bangkok, and declared by all to be 'insufficient for a dress', was ivory-coloured Thai silk, with two bands of delicate colour, one dove-grey and the other *café au lait*, woven into either end. From this, Sir Christopher designed and made a princess-line dress, doing all the cutting, machining and hand-finishing himself. It had a scoop neckline, no sleeves, and fell to just below the knee, with an eight-inch zip hidden in the side seam. It was lined throughout, and skimmed the figure in a manner both comfortable and elegant. It was a great pleasure to wear as well as being very pleasing to the eye. Unfortunately, the figure outgrew the dress in time, and it was given away. But a long evening skirt of five panels, in chestnut-coloured velvet with cream flowers, also designed and made by Sir Christopher for his wife, is still in frequent use.

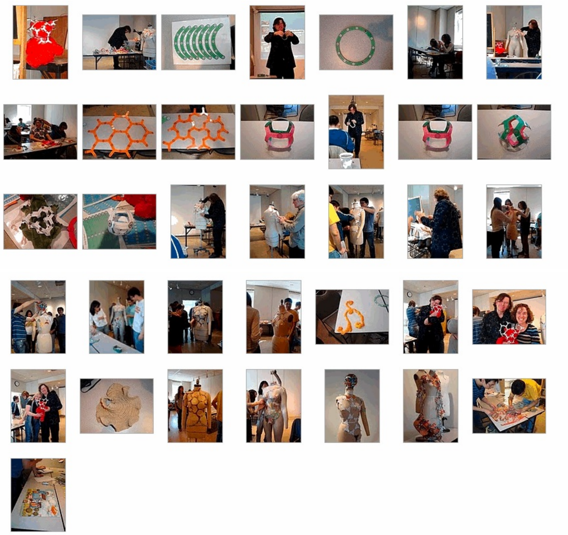
Back to paper, with W. Thurston

Chapter I.5
Pleated Surfaces

I.5.1. Introduction

We now discuss pleated surfaces, which are a basic tool in Thurston's analysis of hyperbolic structures on 3-manifolds. See Section 8.8 of Thurston (1979); there, pleated surfaces are called uncrumpled surfaces.

Recall from definition under Section I.1.3.3 (*Isometric map*) that an isometric map takes rectifiable paths to rectifiable paths of the same length.



William Thurston

William Thurston

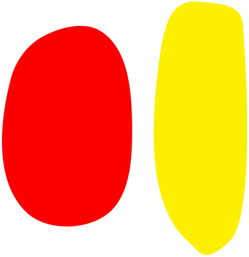
William Thurston

William Thurston

Thurston on Miyake Fashion Show, Paris 2010

with Dai Fujiwara, artistic director Miyake

Theorem (Alexandrov Pogorelov 1970) : *Start with two convex domains with the same perimeter. Glue them along their boundaries (respecting the arc lengths). One gets a convex body in space which is rigid.*



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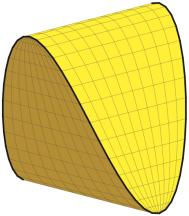
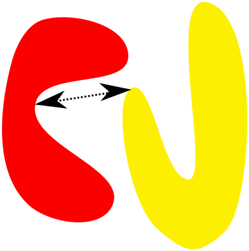


Theorem (Alexandrov Pogorelov 1970) : *The same is true if the two domains are not convex but if the sum of the two curvatures at points which are identified is positive.*



Playing with Surfaces: Spheres, Monkey Pants, and Zippergons

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Abstract

We describe a process, inspired by clothing design, of smoothing an octahedron to form a round sphere. This process can be adapted to construct many different surfaces out of paper and craft foam.

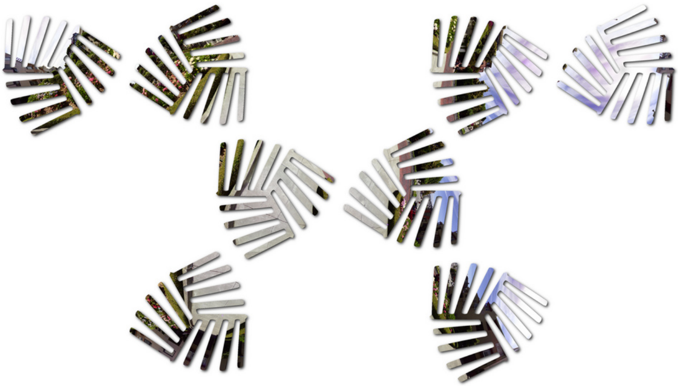
Introduction

(a) *Paper and tape model*

(b) *Pattern*

Figure 1: *Octahedral Sphere*

Kelly Delp, William Thurston, Bridges Coimbra Conference Proceedings (2011), 1-8.



Flicker David Swart

(a) *Intermediate octahedral sphere* (b) *Tetrahedral sphere* (c) *Monkey pants*

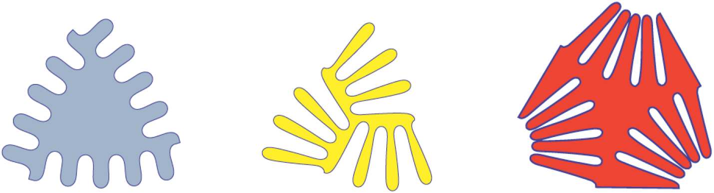


Figure 5: *Paper Models*

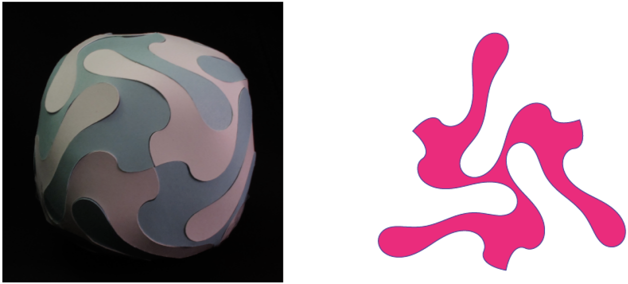
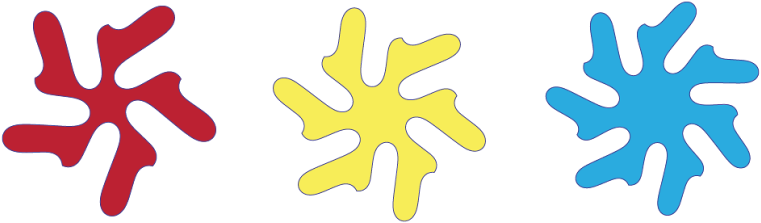


Figure 6: *Octahedral sphere II*

Zippergons



(a) 120° *pentagon* (b) 120° *hexagon* (c) 120° *heptagon*

Figure 7: *Three per vertex system*

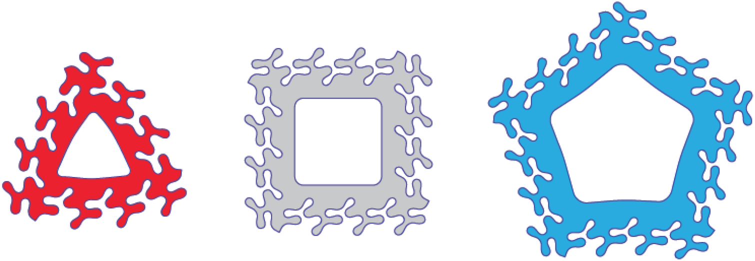
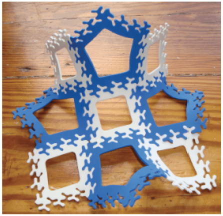


Figure 9: *Improved Zippergons*

(a) *Cuboctahedron*

(b) *Icosadodecahedron*



(c) *Negative curvature*

Figure 10: *Tapeless Zippergon constructions*

Brazuca : FIFA World cup 2014

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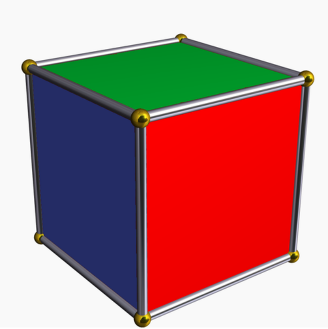
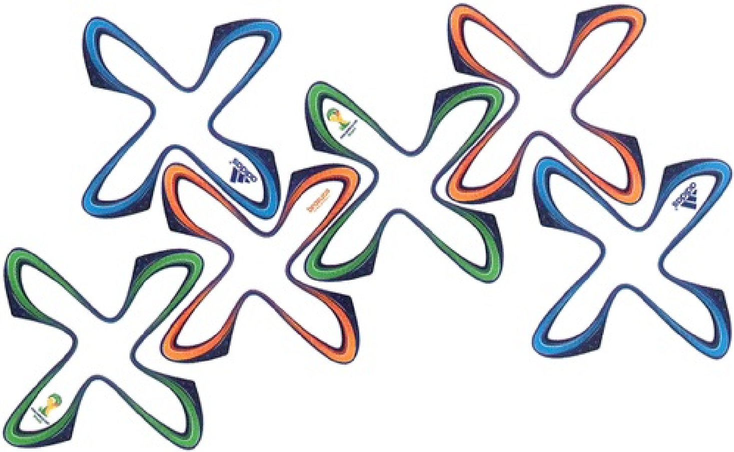
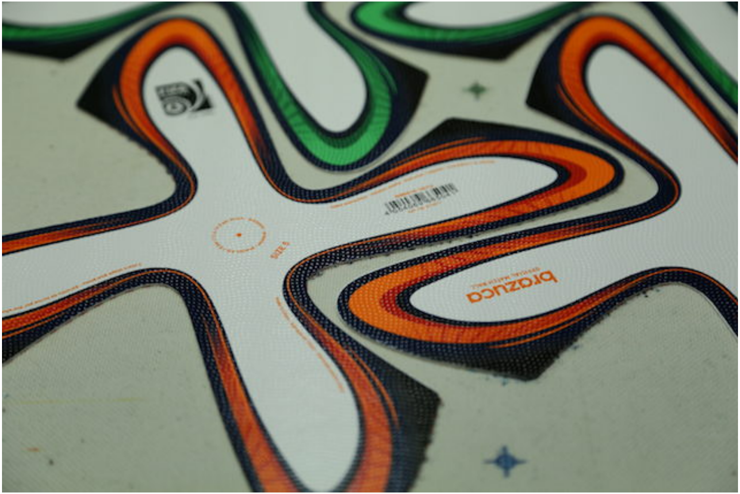


Brazuca : FIFA World cup 2014



Brazuca : FIFA World cup 2014



Brazuca : FIFA World cup 2014

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William Thurston

Bill Wingell for The New York Times