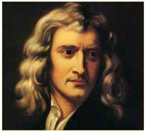
The work of Artur Avila

Étienne Ghys, CNRS ENS Lyon, France

Newton :



You are given an ordinary differential equation and your task is to find its solutions.

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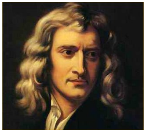

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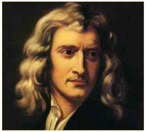
Poincaré :

Newton :



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You are given an ordinary differential equation and your task is to say something about its solutions.

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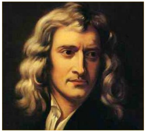

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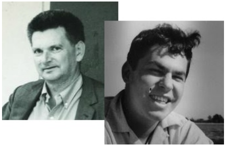

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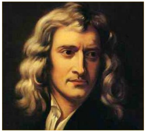

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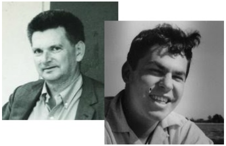

You are not given an ordinary differential equation and your task is to say something about its solutions.

What does a typical dynamical system look like ?


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Institut de Mathématiques de Jussieu

(UMR 7586)

Deux nouveaux docteurs de notre institut, Daniel HOEHENER et Miquel OLIU-BARTON (encadrés respectivement par Hélène Frankowska, par Sylvain Sorin) viennent d'obtenir les deux prix de thèse de PGMO (Programme Gaspard Monge pour l'optimisation et la recherche opérationnelle).

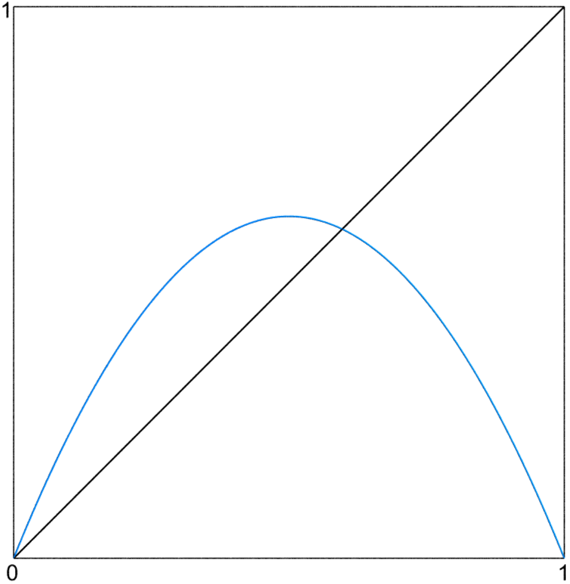
Les Projets de l'Institut

Analyse algébrique	Combinatoire et Optimisation	Histoire des sciences mathématiques
Analyse complexe et géométrie	Formes automorphes	Logique mathématique
Analyse fonctionnelle	Géométrie et dynamique	Théorie des nombres
Algèbres d'opérateurs	Groupes, représentations et géométrie	Topologie et géométrie algébriques

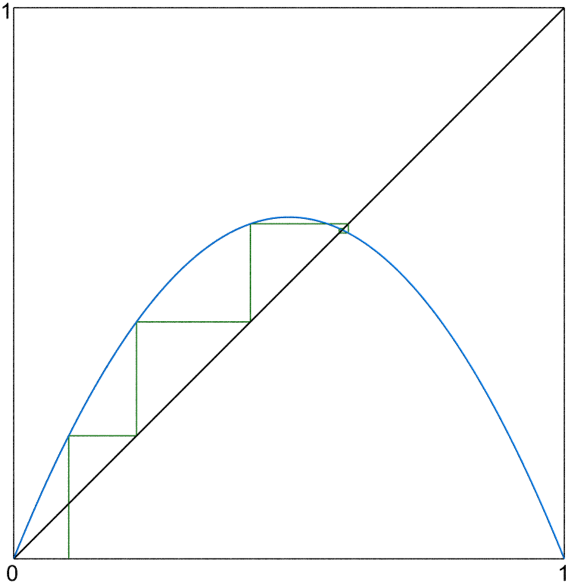
Activités scientifiques au sein de l'Institut

Séminaires	Stage : les étapes clef de la carrière en mathématiques	Journal de l'Institut

Dynamics in dimension 1



Unimodal map



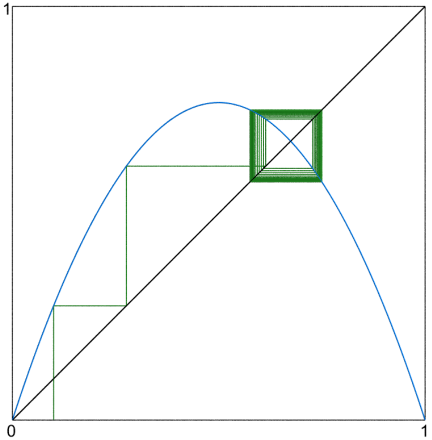
Orbit $\{f^n(x)\}$

Theorem (Avila, de Melo, Lyubich, 2003 + Avila, Moreira, 2005) :

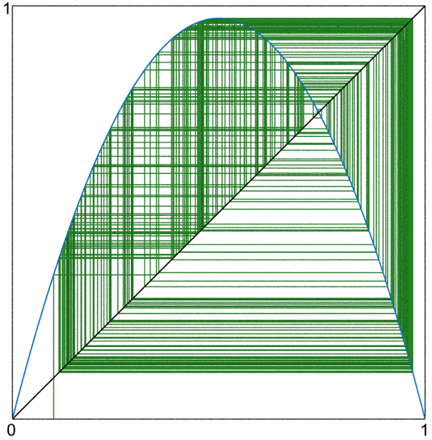
In a non-trivial real analytic family f_λ ($\lambda \in \Lambda \subset \mathbf{R}^N$) of unimodal maps, there is a dichotomy :

For almost every λ, the map f_λ is either Regular or Stochastic.

Regular : Almost every orbit (w.r.t. Lebesgue) converges to an attracting cycle.



Stochastic : there is an absolutely continuous invariant measure μ such that almost every orbit in $[0, 1]$ (w.r.t. Lebesgue) is distributed according to μ, or converges to a cycle.



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or
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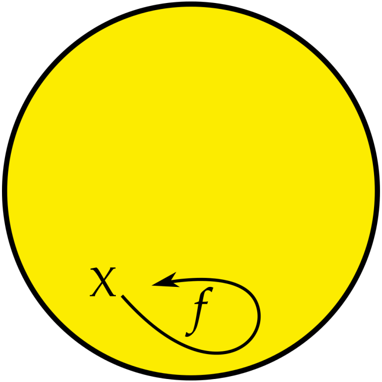
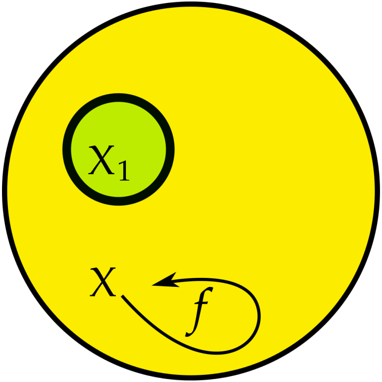
Misha Lyubich :

"We have reached a full probabilistic understanding of real analytic unimodal dynamics, and Artur Avila has been the key player in the final stage of the story".



Renormalization



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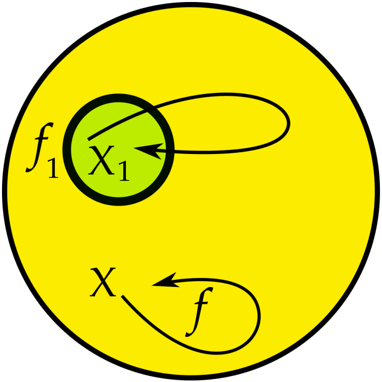
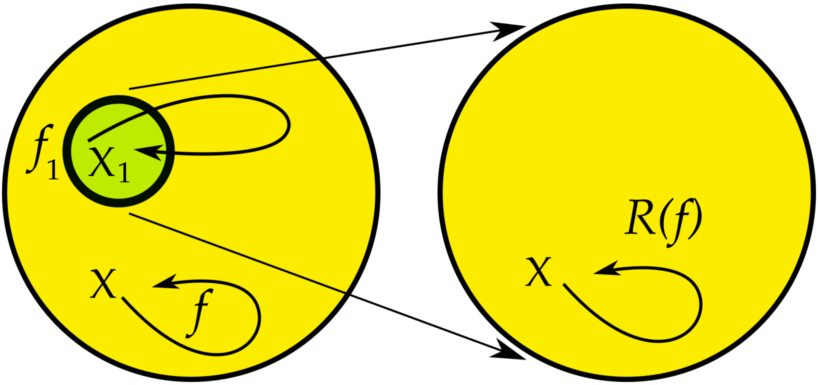


Renormalization



Renormalization

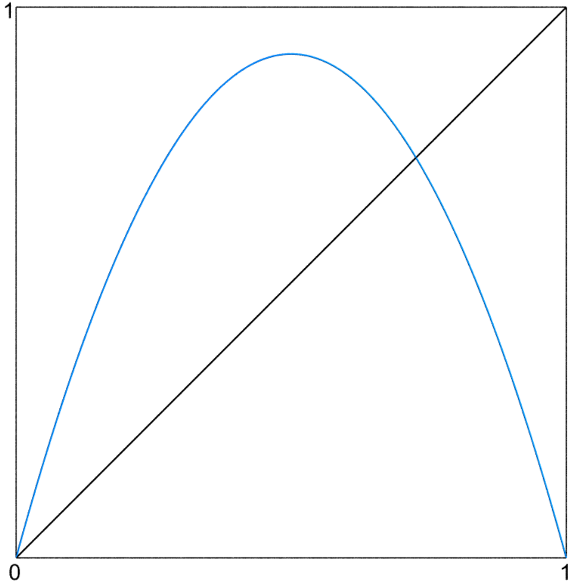
Adrien Douady :
"We first plough in the dynamical plane and then harvest in parameter plane"

Dynamical plane

(Van Gogh)

Parameter plane

(Pissaro)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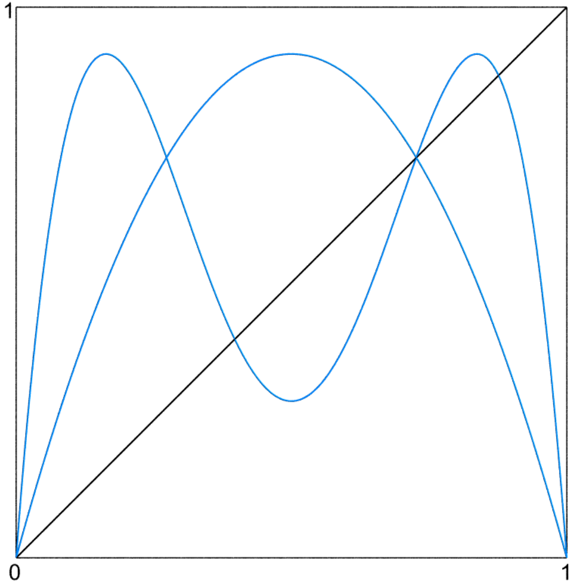
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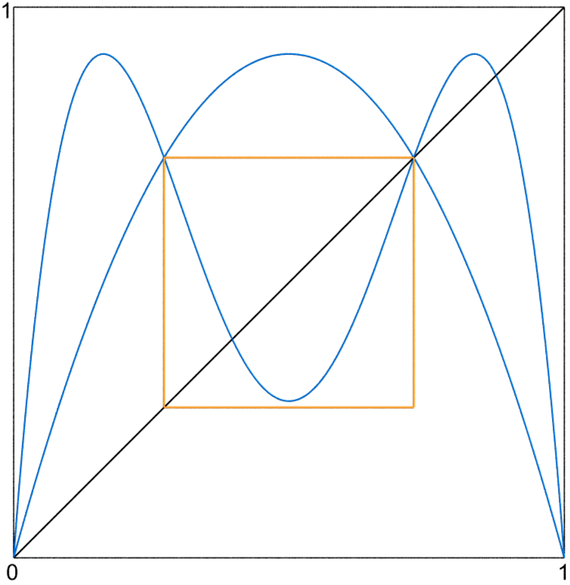
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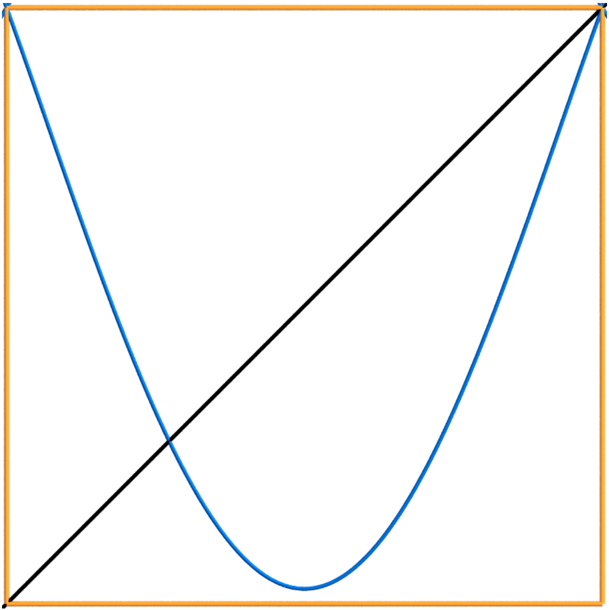
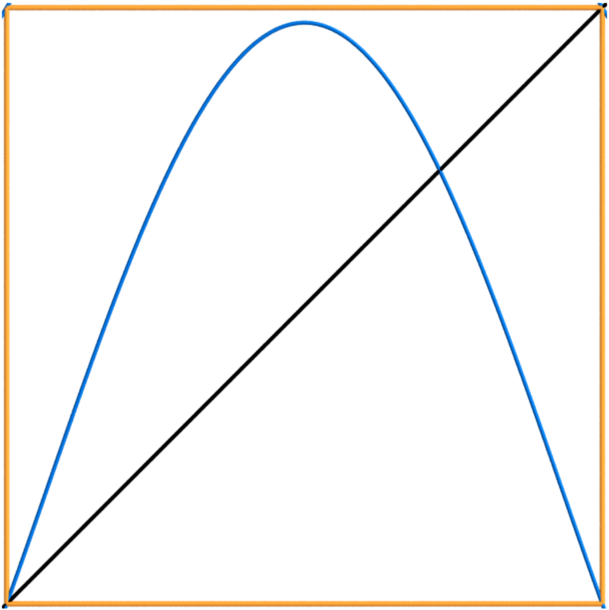
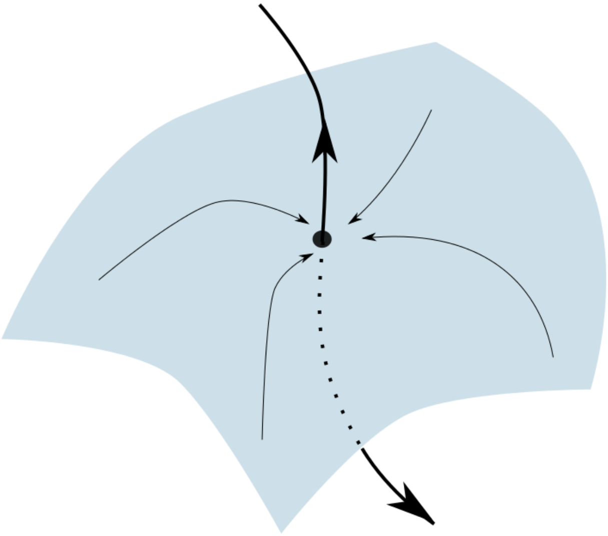
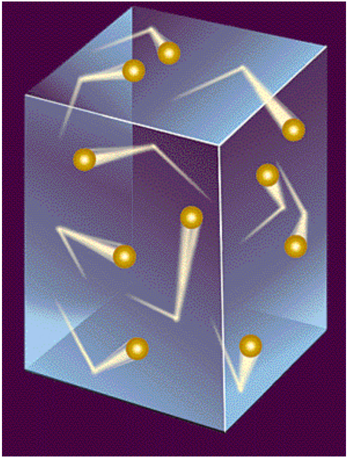


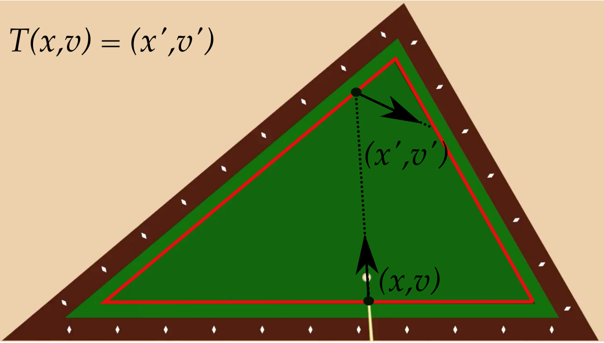




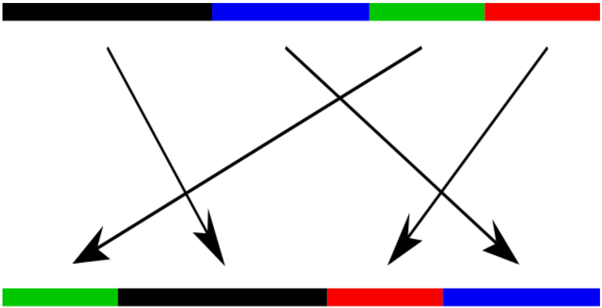
Renormalization on the space of unimodal maps

Billiards

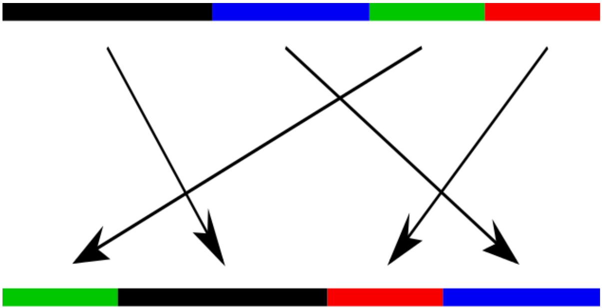




$T(x,v) = (x',v')$
(x',v')
(x,v)



Theorem (Avila, Forni, 2007) :
Almost all (irreducible) interval exchange transformations are weakly mixing.



$f : X \to X$ preserves a probability measure μ.

— Mixing : $\lim_{n\to\infty} \mu(A \cap f^n(B)) = \mu(A)\mu(B)$.

— Weakly mixing : $\lim_{n\in E\to\infty} \mu(A \cap f^n(B)) = \mu(A)\mu(B)$.

($E \subset \mathbf{Z}$ has density 1, i.e. $\lim_{k\to\infty} \frac{1}{k}\sharp(\{1, ..., k\} \cap E) = 1$).

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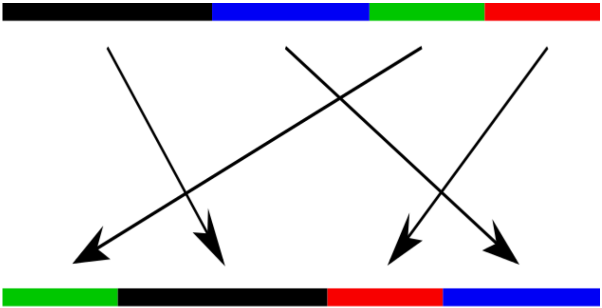

Quasi-periodic Schrödinger operators

$l^2(\mathbf{Z}) = \{\psi : \mathbf{Z} \to \mathbf{C} \quad | \quad \sum_n |\psi(n)|^2 < \infty\}$

Schrödinger equation : $i\frac{\partial \psi}{\partial t} = H\psi$

$H\psi(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$

Spectrum : $\sigma(H) = \{E \in \mathbf{R} \quad | \quad (H - E.\mathrm{Id})^{-1} \quad \text{does not exist}\}$

Spectral measure : $\langle \psi, g(H)\psi \rangle = \int g \, d\mu_\psi$

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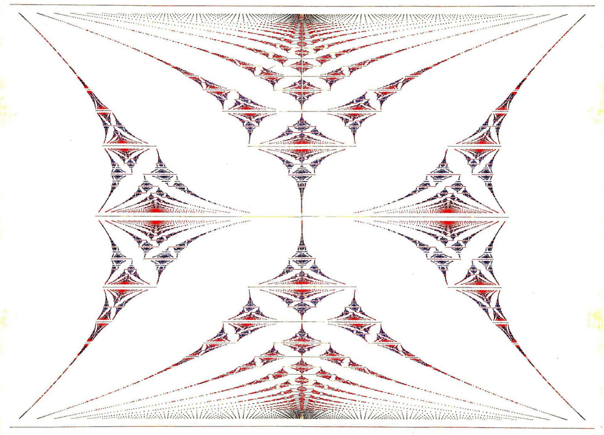
Spectral measure

- absolutely continuous $\implies$ the particle travels freely.
- singular continuous $\implies$ the particle travels somewhat.
- pure point $\implies$ does not travel.

Almost Mathieu operator :

$$V(n) = 2\lambda \cos(2\pi n\alpha)$$



Hofstadter's butterfly

Theorem (Avila, Jitomirskaya, 2009) : Ten Martini conjecture.

For all $\lambda \neq 0$*, and all irrational* α*, the spectrum* $\sigma_{\lambda,\alpha}$ *is a Cantor set.*

Theorem (Avila, Krikorian, 2006) : Aubry André's conjecture.

$Leb(\sigma_{\lambda,\alpha}) = 4|1 - |\lambda||$

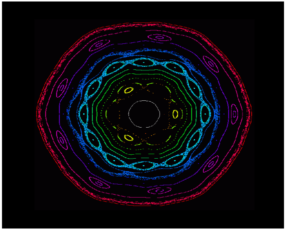
Theorem (Avila-Jitomirskaya 2009, Avila-Damanik 2008, Avila 2009) :

For all irrational α *and* $|\lambda| < 1$*, the spectrum is purely absolutely continuous.*

Conservative dynamics

On a compact C^∞ manifold.

Well known and easy :
Any C^1 diffeomorphism can be C^1 approximated by C^∞ diffeomorphisms.



On a compact C^∞ manifold equipped with a C^∞ volume form.

Theorem (Avila, 2010) :
Any C^1 volume preserving diffeomorphism can be C^1 approximated by C^∞ diffeomorphisms which are volume preserving.