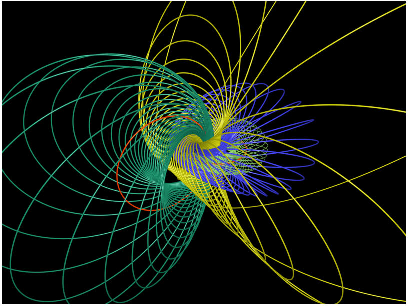
A guided tour of the seventh dimension

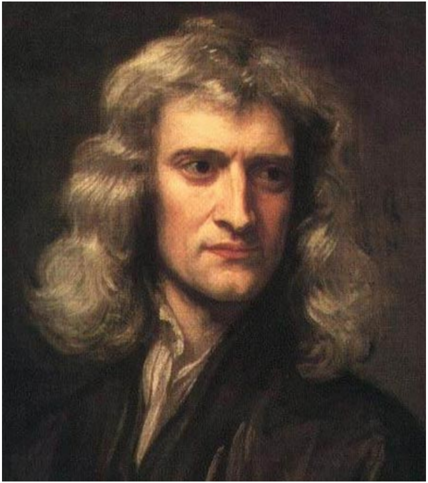


Étienne Ghys

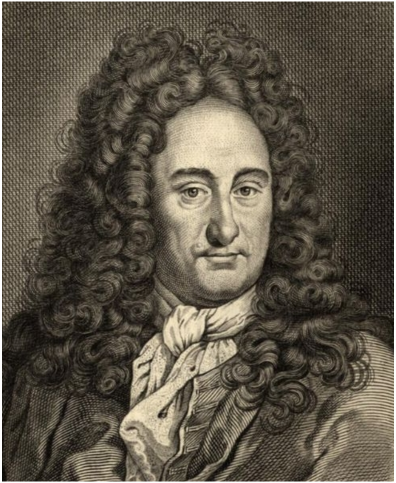
Unité de Mathématiques Pures et Appliquées

UMR 5669 CNRS – École Normale Supérieure de Lyon

Isaac Newton (1643-1727)

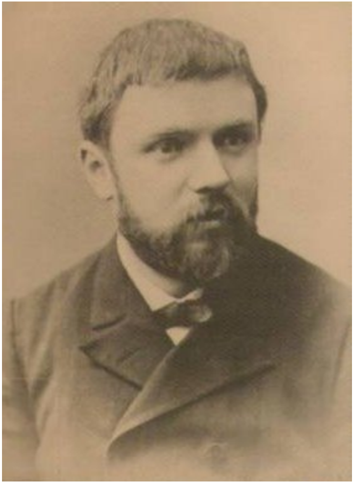


Gottfried Wilhelm Leibniz (1646-1716)



Immanuel Kant (1724-1804)

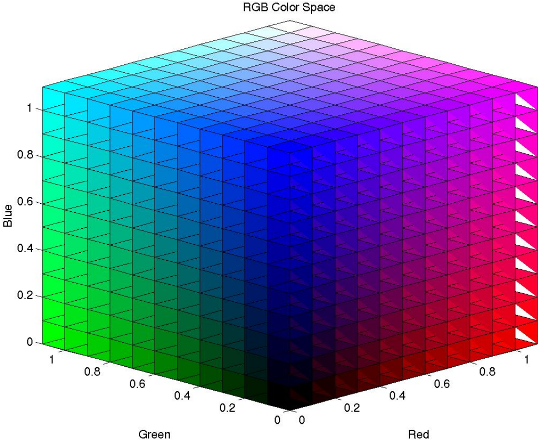
Henri Poincaré (1854-1912)



Poincaré

Poincaré

Color space RGB



Mantis shrimp

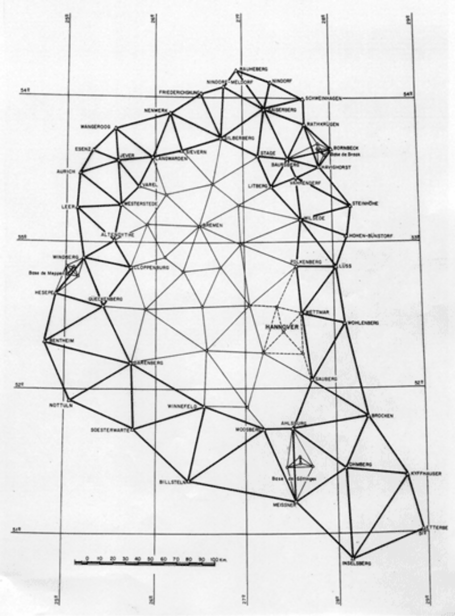
Poincaré

"There is a topology of more than 3 dimensions. I do not say that this is an easy science; I have devoted too much effort to it not to have taken account of the difficulties which are encountered in it. But nevertheless, this science is possible and it does not rest exclusively on analysis. It could not be pursued successfully without a continual appeal to intuition. Therefore, there is surely an intuition about the continua of more than three dimensions and if it demands more sustained attention it is doubtless a matter of habits [...]
Do we not see in our high-schools pupils who do well in plane geometry but "who cannot visualize space" ? It is not their intuition of three dimensional space which is lacking, but they are not in the habit of using it and they need to make an effort to do so. "

Poincaré

"I shall conclude that there is in all of us an intuitive notion of the continuum of any number of dimensions whatever because we possess the capacity to construct a physical and mathematical continuum ; and this capacity exists in us before any experience. [...] It is the exterior world, it is the experience which induces us to make use of it."

Bernhard Riemann (1826-1866)



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FRIEDERICHSKUNG
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SIEVERN
BORNBECK
STAGE
Base de Brank
AURICH
LITBERG
VAREL
STEINHÖHE
BREMEN
HOHEN-BÜNSTORF
LÜSS
WINDBERG
CLOPPENBURG
Base de Meppen
HESEPE
HANNOVER
WOHLENBERG
NOTTULN
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SOESTERWARTE
BROCKEN
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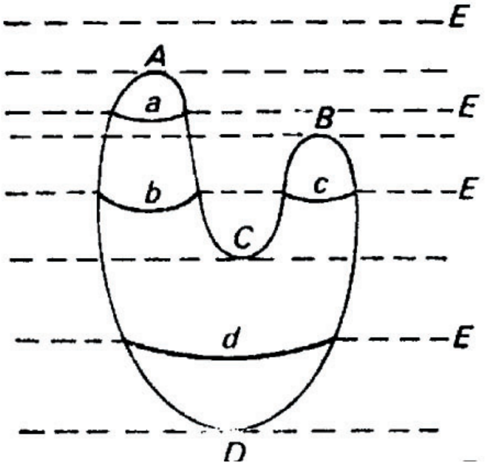
August Ferdinand Moebius (1790-1868)



August Ferdinand Moebius (1790-1868)

A. F. Möbius.

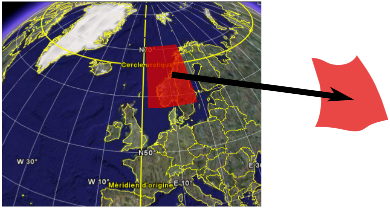
Milnor's conjecture

For every convex domain $C \subset \mathbf{S}^2$ contained in some hemisphere, there is a map $f : C \to \mathbf{R}^2$ which is such that

$$1 \leq \frac{\mathrm{distance}_{\mathbf{R}^2}(f(x), f(y))}{\mathrm{distance}_{\mathbf{S}^2}(x, y)} \leq \exp(\epsilon)$$

with

$$\epsilon = \frac{\mathrm{area}(C)}{\mathrm{area}(\mathbf{S}^2)}$$



Alicia Boole (1860-1840)

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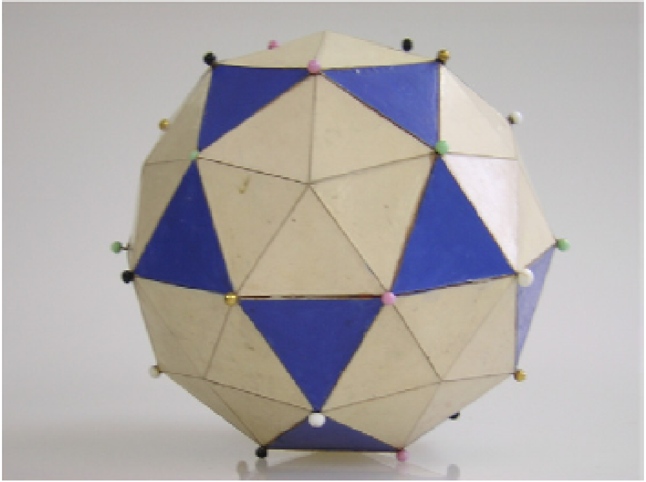


Poincaré

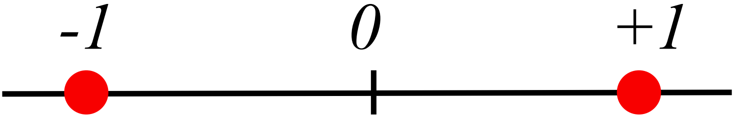
“In order to define the continum of n dimensions, we first of all have the analytic definition: a continum of n dimensions is a set of n coordinates, that is, a set of n quantities capable of varying independently one from the other [...]. This definition, flawless from the point of view of mathematics, nevertheless could not be entirely satisfactory to us. In a continum the diverse coordinates are not, so to speak, juxtaposed one to the other; they are linked among themselves so as to form the various aspects of a whole. At each instant in the study of space, we carry out what is called a change of coordinates.” [...]

Poincaré

"This definition minimizes the importance of the intuitive origin of the notion of continum and the rich ideas which this notion contains. It recurs in the type of definitions which have become so frequent in mathematics since the tendency to "arithmetize" this science. These definitions could not satisfy the philosopher. I do not mean that this "arithmetization" of mathematics is undesirable; I say that this is not everything."

Seventh dimension

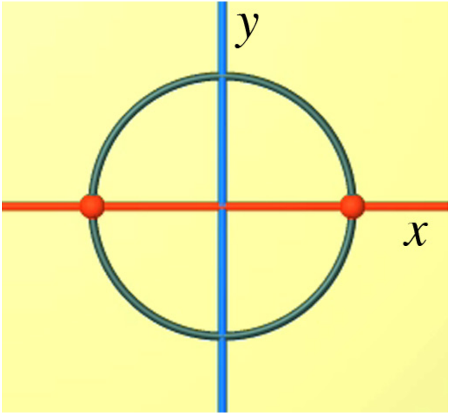
$\mathbf{S}^0$: Dimension zero...



$$\mathbf{S}^0 = \{x \in \mathbf{R} |\ |x| = 1\}$$

$$\mathbf{S}^0 = \{-1, +1\}$$

$\mathbf{S}^1$: Dimension one...



$$\mathbf{S}^1 = \{(x, y) \in \mathbf{R}^2 \mid \sqrt{x^2 + y^2} = 1\}$$

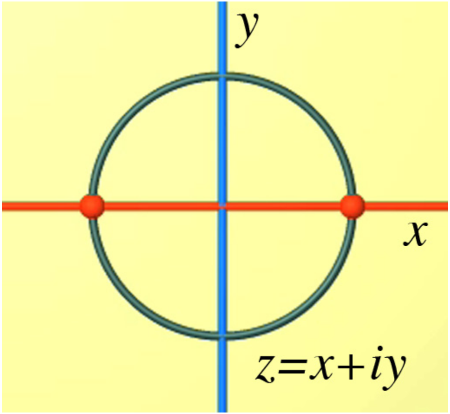
Dimension one...

Complex numbers: $\mathbf{C} = \{z = x + iy\}$ with $i^2 = -1$

Modulus; $|z| := \sqrt{x^2 + y^2}$

modulus multiplicative : $|z_1 z_2| = |z_1||z_2|$

$\mathbf{S}^1$: Dimension one...



$$\mathbf{S}^1 = \{z = x + iy \in \mathbf{C} \mid |z| = 1\}$$

$\mathbf{S}^2$: Dimension two...

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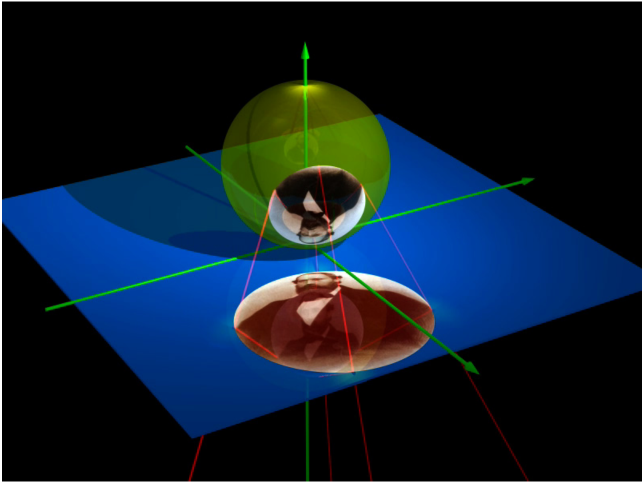


Stereographic projection

Quaternions

$$\mathbf{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k\}$$

$$ij = -ij = k; \quad jk = -kj = i; \quad ki = -ik = j$$

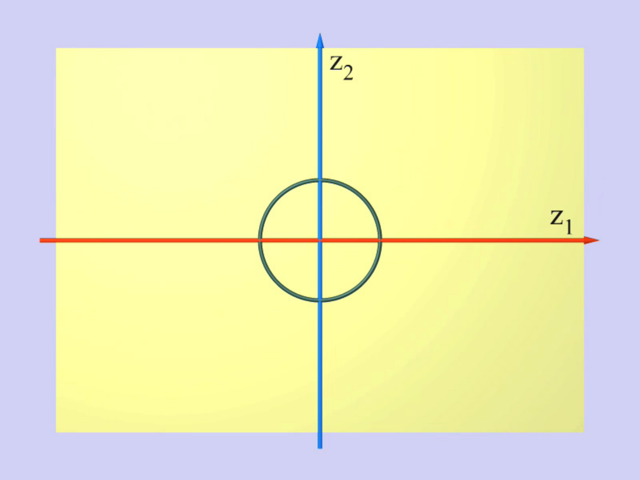
Quaternions

$$\mathbf{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k\}$$

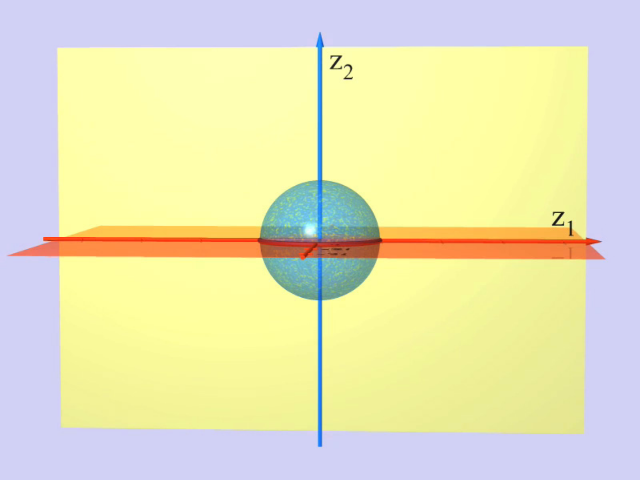
$$|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

$$|q.q'| = |q||q'|$$

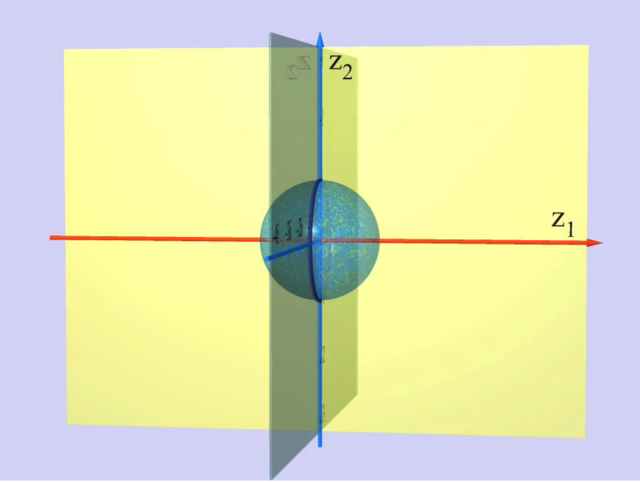
Dimension three : $\mathbf{S}^3 \subset \mathbf{R}^4 = \mathbf{H}$



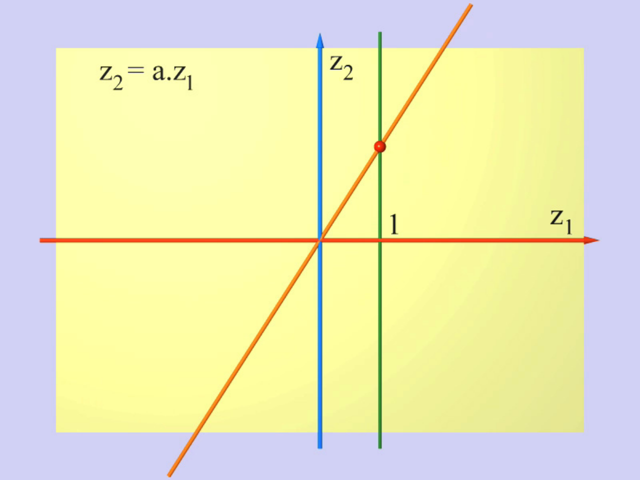
Dimension three : $\mathbf{S}^3 \subset \mathbf{R}^4 = \mathbf{H}$



Dimension three : $\mathbf{S}^3 \subset \mathbf{R}^4 = \mathbf{H}$



Dimension three : $\mathbf{S}^3 \subset \mathbf{R}^4 = \mathbf{H}$



Hopf fibration

Hopf fibration

$$\pi : \mathbf{S}^3 \longrightarrow \mathbf{S}^2$$

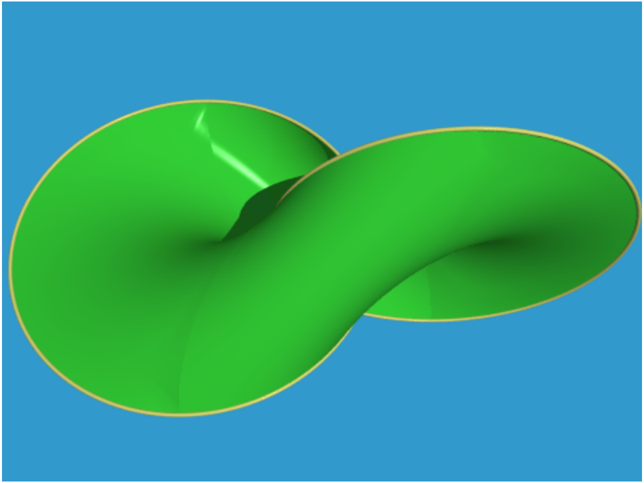
$$\pi^{-1}(\star) \simeq \mathbf{S}^1$$

Hopf fibration

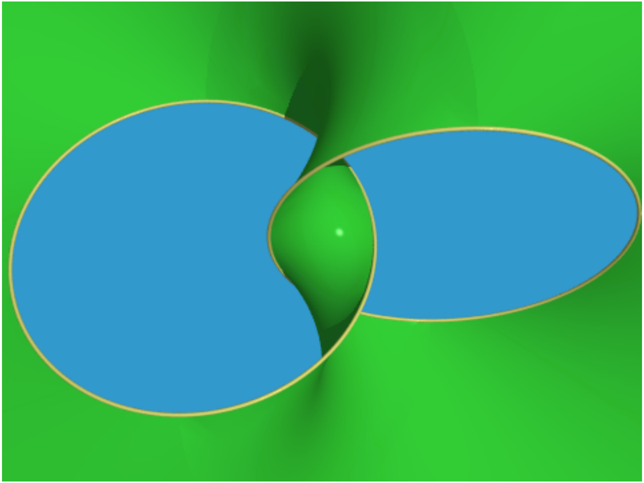
$xy = 0$

$$(x, y) \in \mathbf{S}^3 \setminus \{xy = 0\} \mapsto \frac{xy}{|xy|} \in \mathbf{S}^1$$

$xy = 0$



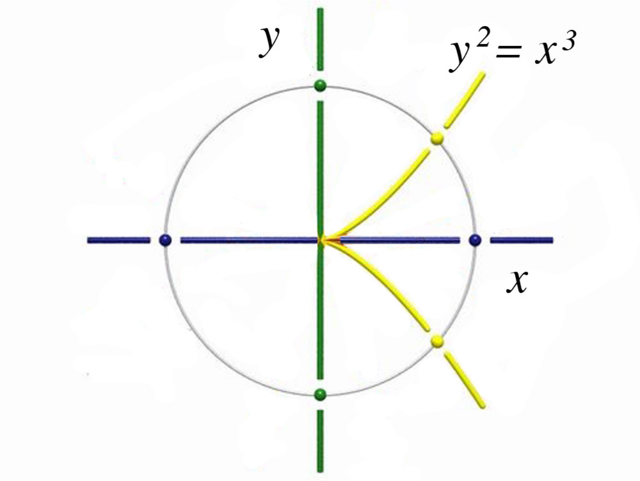
$xy = 0$



$xy(x-y)=0$

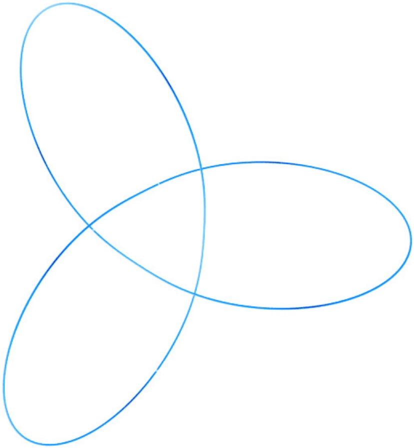
$$(x,y) \in \mathbf{S}^3 \setminus \{xy(x-y)=0\} \mapsto \frac{xy(x-y)}{|xy(x-y)|} \in \mathbf{S}^1$$

$xy(x-y)=0$



y
$y^2 = x^3$
x

$y^2 = x^3$



$y^2 = x^3$

$$(x, y) \in \mathbf{S}^3 \setminus \{y^2 = x^3\} \mapsto \frac{y^2 - x^3}{|y^2 - x^3|} \in \mathbf{S}^1$$

Milnor fibration

$y^4 - 2y^2x^3 + x^6 - x^7 - 4yx^5 = 0$

Octonions

$$\mathbf{O} = \{\mathfrak{o} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7\}$$

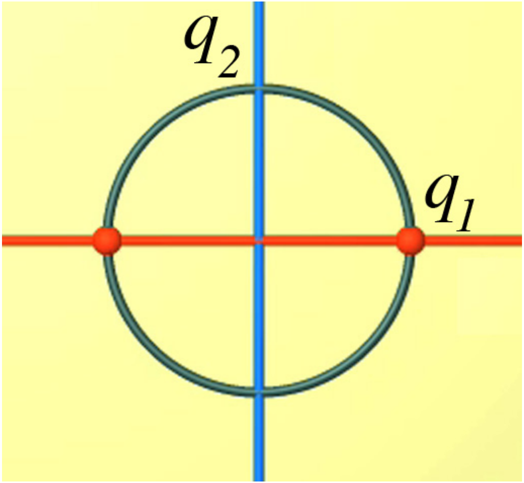
	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Octonions

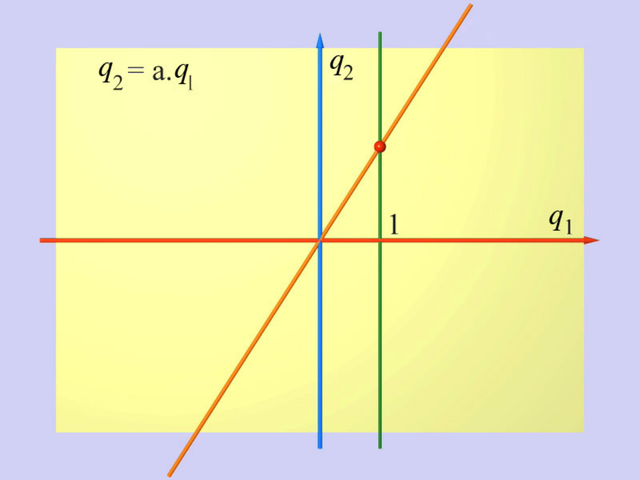
$$|\mathfrak{o}| = \sqrt{\sum_{0}^{7} x_l^2}$$

$$|\mathfrak{o}_1 \mathfrak{o}_2| = |\mathfrak{o}_1|\,|\mathfrak{o}_2|$$

The seven sphere $\mathbf{S}^7$



The seven sphere $\mathbf{S}^7$



Hopf fibration

$$\pi : (q_1, q_2) \in \mathbf{S}^7 \mapsto q_2 q_1^{-1} \in \mathbf{H} \cup \{\infty\} \simeq \mathbf{S}^4$$

$$\pi^{-1}(\star) \simeq \mathbf{S}^3$$

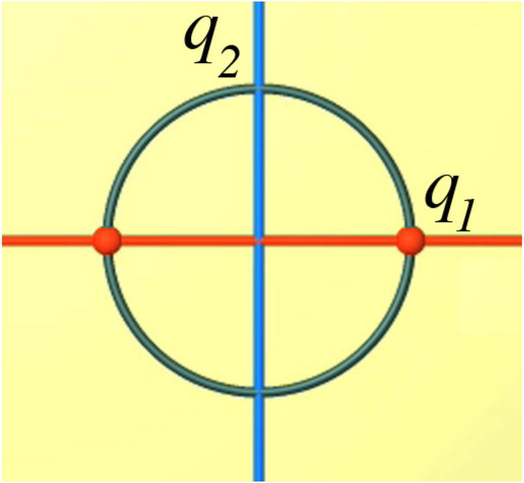
Theorem (Kervaire, Bott-Milnor) (1958) :

Suppose there is a bilinear multiplication

$$m : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$$

with no zero divisor,
(i.e. if $m(x, y) = 0$*, then* $x = 0$ *or* $y = 0$*).*

Then the dimension n *is* $0, 1, 2, 4$ *or* 8*.*


q_2
q_1

Decomposition

$$|q_1|^2 + |q_2|^2 = 1$$

$$\text{Part1}: \; |q_2| \leq \frac{\sqrt{2}}{2} \leq |q_1| \qquad \text{Part2}: \; |q_1| \leq \frac{\sqrt{2}}{2} \leq |q_2|$$

Decomposition

$$|q_1|^2 + |q_2|^2 = 1$$

$$\mathrm{Part1}: \ |q_2| \leq \frac{\sqrt{2}}{2} \leq |q_1| \qquad \mathrm{Part2}: \ |q_1| \leq \frac{\sqrt{2}}{2} \leq |q_2|$$

Decomposition

Part One

$$x := q_2 q_1^{-1}; \qquad y := q_1/|q_1|$$

$$|x| \leq 1; \quad |y| = 1$$

$$\mathbf{B}^4 \times \mathbf{S}^3$$

Part two

$$x' := \overline{q_1 q_2^{-1}}; \qquad y' := q_2/|q_2|$$

$$|x'| \leq 1; \quad |y'| = 1$$

$$\mathbf{B}^4 \times \mathbf{S}^3$$

On the boundary, $|x| = |y| = 1$:

$$(x, y) \simeq (x, xy)$$

Part two

$$x' := \overline{q_1 q_2^{-1}}; \qquad y' := q_2/|q_2|$$

$$|x'| \leq 1; \quad |y'| = 1$$

$$\mathbf{B}^4 \times \mathbf{S}^3$$

On the boundary, $|x| = |y| = 1$:

$$(x, y) \simeq (x, xy)$$

Recipe for $\mathbf{S}^7$

Ingredients : Two 4 balls $\mathbf{B}^4_1$ and $\mathbf{B}^4_2$ and two 3-spheres $\mathbf{S}^3_1$ and $\mathbf{S}^3_2$.

1. Form the products $\mathbf{B}^4_1 \times \mathbf{S}^3_1$ and $\mathbf{B}^4_2 \times \mathbf{S}^3_2$.
2. Glue the boundaries.
   Glue $(x, y) \in \partial\mathbf{B}^4_1 \times \mathbf{S}^3_1$ with $(x, xy) \in \partial\mathbf{B}^4_2 \times \mathbf{S}^3_1$.

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Recipe for $\mathbf{S}^3$

Ingredients : Two 2 balls $\mathbf{B}^2_1$ and $\mathbf{B}^2_2$ and two 1-spheres $\mathbf{S}^1_1$ and $\mathbf{S}^1_2$.

1. Form the products $\mathbf{B}^2_1 \times \mathbf{S}^1_1$ and $\mathbf{B}^2_2 \times \mathbf{S}^1_2$.
2. Glue the boundaries.
   Glue $(x, y) \in \partial\mathbf{B}^2_1 \times \mathbf{S}^1_1$ with $(x, xy) \in \partial\mathbf{B}^2_2 \times \mathbf{S}^1_1$.

Recipe for $\mathbf{S}^3$

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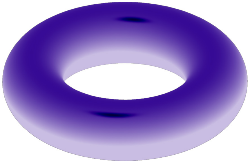
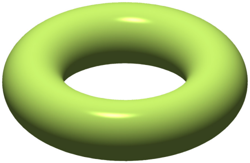
   Glue $(x, y) \in \partial\mathbf{B}^2_1 \times \mathbf{S}^1_1$ with $(x, xy) \in \partial\mathbf{B}^2_2 \times \mathbf{S}^1_1$.

Two solid tori

Recipe for an exotic Σ^7

Ingredients : Two 4 balls $\mathbf{B}_1^4$ and $\mathbf{B}_2^4$ and two 3-spheres $\mathbf{S}_1^3$ and $\mathbf{S}_2^3$.

1. Form the products $\mathbf{B}_1^4 \times \mathbf{S}_1^3$ and $\mathbf{B}_2^4 \times \mathbf{S}_2^3$.
2. Glue the boundaries.
   Glue $(x, y) \in \partial\mathbf{B}_1^4 \times \mathbf{S}_1^3$ with $(x, x^2yx^{-1}) \in \partial\mathbf{B}_2^4 \times \mathbf{S}_1^3$.

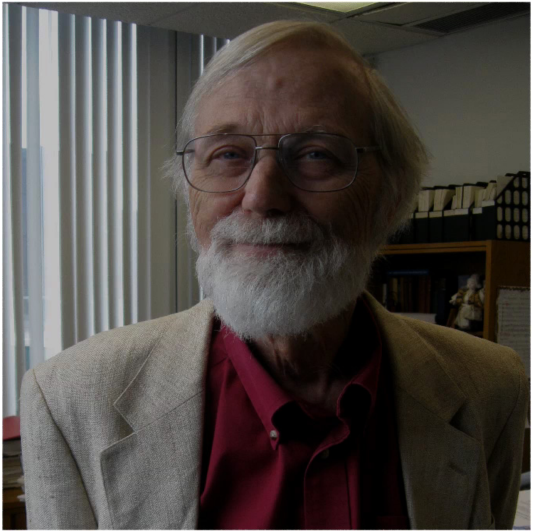
Ingredients : Two 4 balls $\mathbf{B}_1^4$ and $\mathbf{B}_2^4$ and two 3-spheres $\mathbf{S}_1^3$ and $\mathbf{S}_2^3$.

1. Form the products $\mathbf{B}_1^4 \times \mathbf{S}_1^3$ and $\mathbf{B}_2^4 \times \mathbf{S}_2^3$.
2. Glue the boundaries.
   Glue $(x, y) \in \partial\mathbf{B}_1^4 \times \mathbf{S}_1^3$ with $(x, x^2yx^{-1}) \in \partial\mathbf{B}_2^4 \times \mathbf{S}_1^3$.

Theorem (Milnor) (1956) :

1. Σ^7 *is homeomorphic to* $\mathbf{S}^7$.
2. Σ^7 *is not diffeomorphic to* $\mathbf{S}^7$.

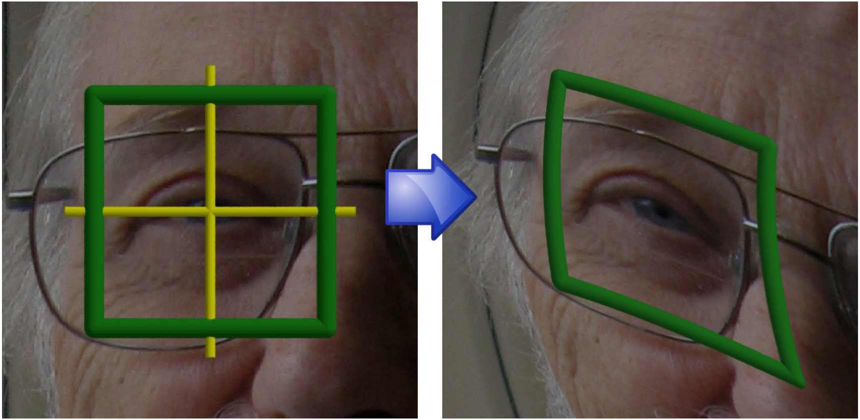
Diffeomorphism



Diffeomorphism

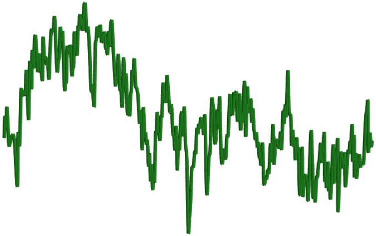
Diffeomorphism

Diffeomorphism



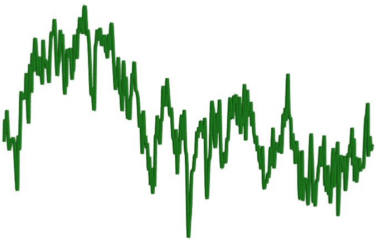
Homeomorphism

A continuous function



$$f_\epsilon(x) := \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(t)dt$$

A continuous function



$$f_\epsilon(x) := \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(t)dt$$

A continuous function

Homeomorphism

Smoothing

Thanks to Jos Leys