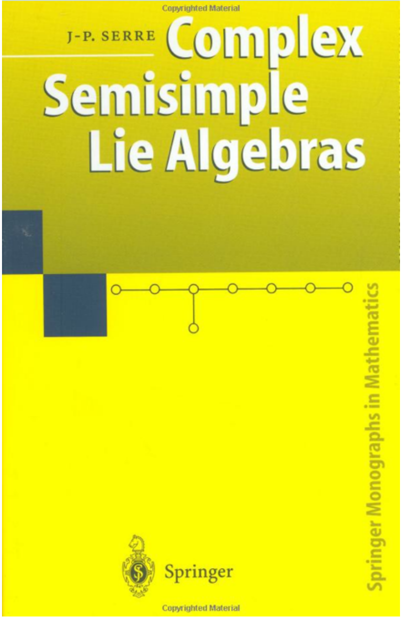
Inner simplicity *vs.* outer simplicity

Étienne Ghys, École Normale Supérieure de Lyon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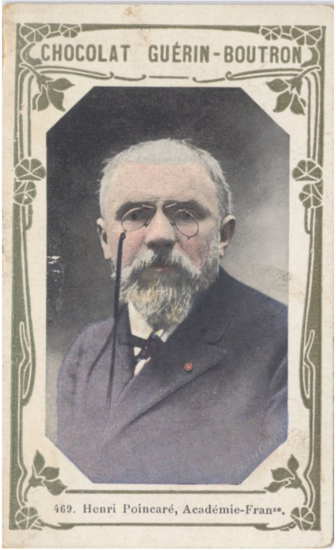


Logic vs. Intuition

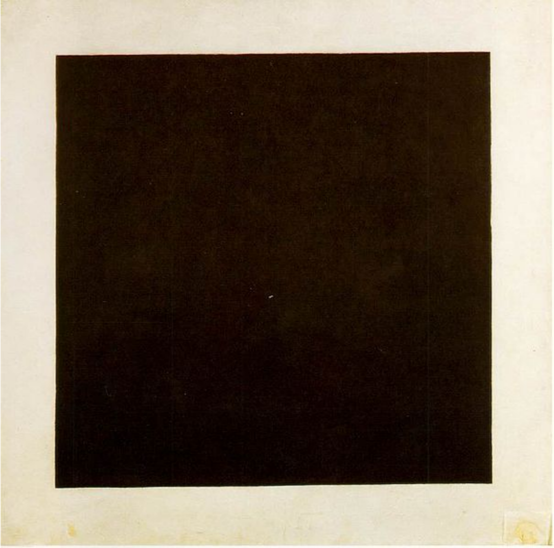
simplicity (n.)late 14c., from Old French *simplicite* (French *simplicité*), from Latin *simplicitatem* (nominative *simplicitas*) "state of being simple," from *simplex* (genitive *simplicis*) "simple".

simplex (adj.) "characterized by a single part," 1590s, from Latin *simplex* "single, simple," from PIE root **sem-* "one, together" (cf. Latin *semper* "always," literally "once for all ;" Sanskrit *sam* "together ;" see same) + **plac-* "-fold." The noun is attested from 1892.

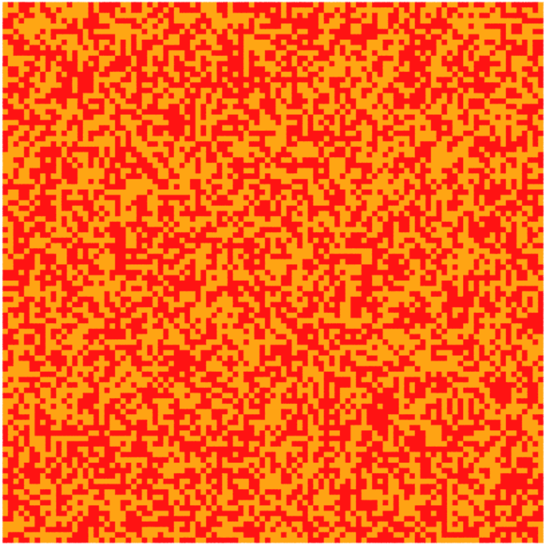
complex (adj.) 1650s, "composed of parts," from French *complexe* "complicated, complex, intricate" (17c.), from Latin *complexus* "surrounding, encompassing," past participle of *complecti* "to encircle, embrace," in transferred use, "to hold fast, master, comprehend," from *com-* "with" (see com-) + *plectere* "to weave, braid, twine, entwine," from PIE **plek-to-*, from root **plek-* "to plait" (see ply). The meaning "not easily analyzed" is first recorded 1715. *Complex sentence* is attested from 1881.

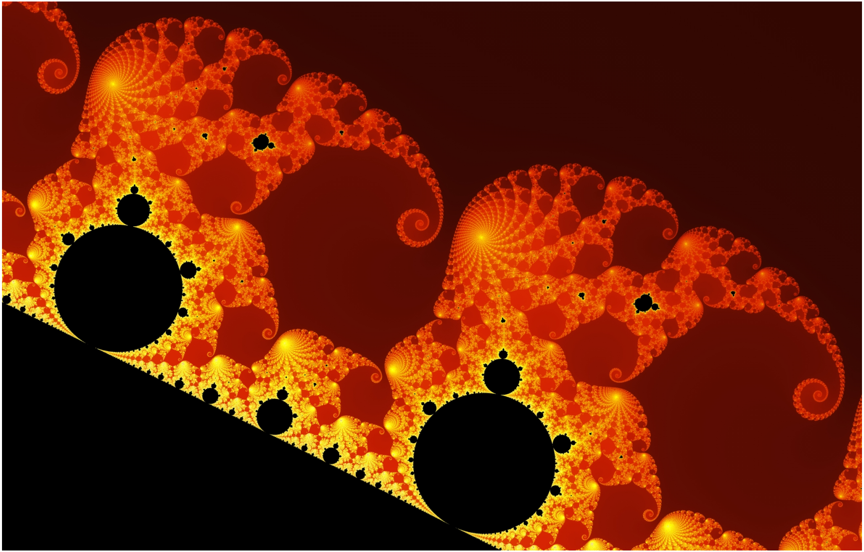
Andrey Kolmogorov

Complexity = Length of the shortest description



Black Square Malevich (1915)





CHAPITRE I

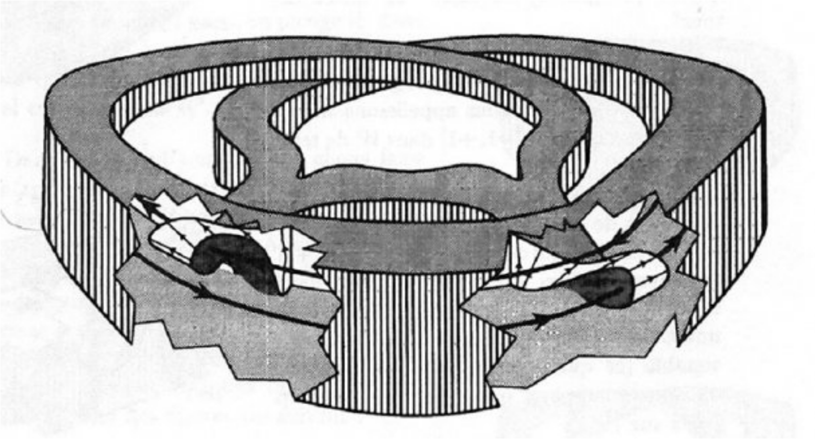
CORPS FINIS

Tous les corps considérés ci-dessous sont supposés commutatifs.

§ 1. Généralités

1.1. *Corps premiers et corps finis.*

L'intersection des sous-corps d'un corps K en est le plus petit sous-corps; il contient l'image canonique de $\mathbf{Z}$, isomorphe en tant qu'anneau intègre à $\mathbf{Z}$ ou à $\mathbf{Z}/p\mathbf{Z}$ avec p premier; il est donc isomorphe, soit à $\mathbf{Q}$, soit au corps $\mathbf{Z}/p\mathbf{Z}$.



Hilbert zeroth problem ?

To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"?

Hilbert zeroth problem ?

Who does not make use of drawings of segments and rectangles enclosed in one another, when it is required to prove with perfect rigor a difficult theorem on the continuity of functions or the existence of points of condensation? Who could dispense with the figure of the triangle, the circle with its center, or with the cross of three perpendicular axes? Or who would give up the representation of the vector field, or the picture of a family of curves or surfaces with its envelope which plays so important a part in differential geometry, in the theory of differential equations, in the foundation of the calculus of variations and in other purely mathematical sciences?

Hilbert zeroth problem ?

The arithmetical symbols are written diagrams and the geometrical figures are graphic formulas; and no mathematician could spare these graphic formulas, any more than in calculation the insertion and removal of parentheses or the use of other analytical signs.

The use of geometrical signs as a means of strict proof presupposes the exact knowledge and complete mastery of the axioms which underlie those figures; and in order that these geometrical figures may be incorporated in the general treasure of mathematical signs, there is necessary a rigorous axiomatic investigation of their conceptual content. Just as in adding two numbers, one must place the digits under each other in the right order, so that only the rules of calculation, *i. e.*, the axioms of arithmetic, determine the correct use of the digits, so the use of geometrical signs is determined by the axioms of geometrical concepts and their combinations.

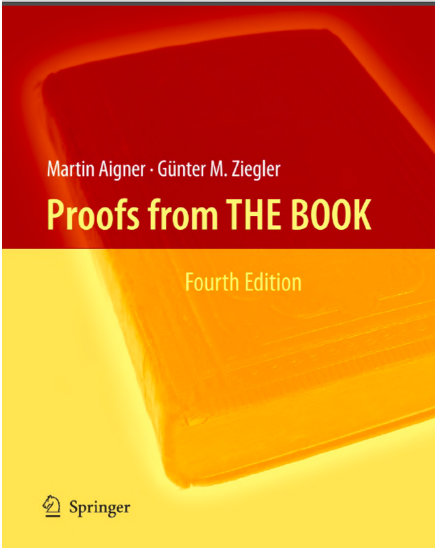
Hilbert zeroth problem ?

The agreement between geometrical and arithmetical thought is shown also in that we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions. On the contrary we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behavior of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry. As an example of an arithmetical theory operating rigorously with geometrical ideas and signs, I may mention Minkowski's work, Die Geometrie der Zahlen. *

Intuition and Poincaré

If you are present at a game of chess, it will not suffice, for the understanding of the game, to know the rules for moving the pieces. That will only enable you to recognize that each move has been made conformably to these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book on mathematics would do if he were a logician only. To understand the game is wholly another matter; it is to know why the player moves this piece rather than that other which he could have moved without breaking the rules of the game. It is to perceive the inward reason which makes of this series of successive moves a sort of organized whole. This faculty is still more necessary for the player himself, that is, for the inventor.

Poincaré (The value of Science, Mathematical creation).


Martin Aigner · Günter M. Ziegler
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In fact, what is mathematical creation? It does not consist in making new combinations with mathematical entities already known. Any one could do that, but the combinations so made would be infinite in number and most of them absolutely without interest. To create consists precisely in not making useless combinations and in making those which are useful and which are only a small minority. Invention is discernment, choice.

How to make this choice I have before explained; the mathematical facts worthy of being studied are those which, by their analogy with other facts, are capable of leading us to the knowledge of a mathematical law just as experimental facts lead us to the knowledge of a physical law. They are those which reveal to us unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another.

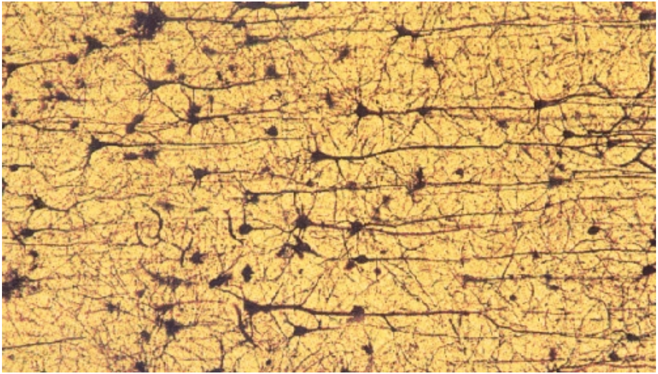
Poincaré (The value of Science, Intuition and logic in mathematics).

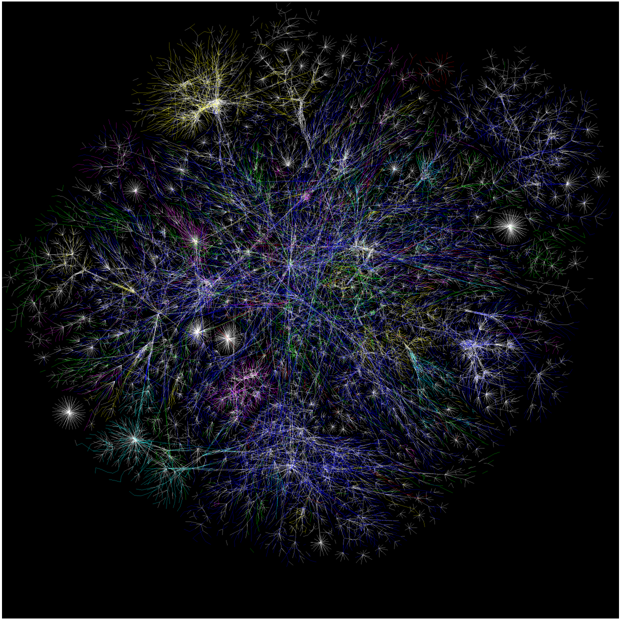
Among all the constructs which can be built up of the materials furnished by logic, choice must be made; the true geometer makes this choice judiciously because he is guided by a sure instinct, or by some vague consciousness of I know not what more profound and more hidden geometry, which alone gives value to the edifice constructed.

To seek the origin of this instinct, to study the laws of this deep geometry, felt, not stated, would also be a fine employment for the philosophers who do not want logic to be all. But it is not at this point of view I wish to put myself, it is not thus I wish to consider the question. The instinct mentioned is necessary for the inventor, but it would seem at first we might do without it in studying the science once created. Well, what I wish to investigate is if it be true that, the principles of logic once admitted, one can, I do not say discover, but demonstrate, all the mathematical verities without making a new appeal to intuition.

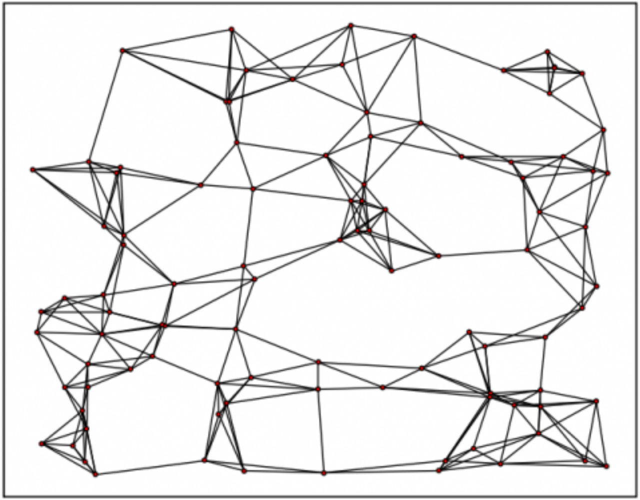
Poincaré (Science and Method, Mathematics and logic).

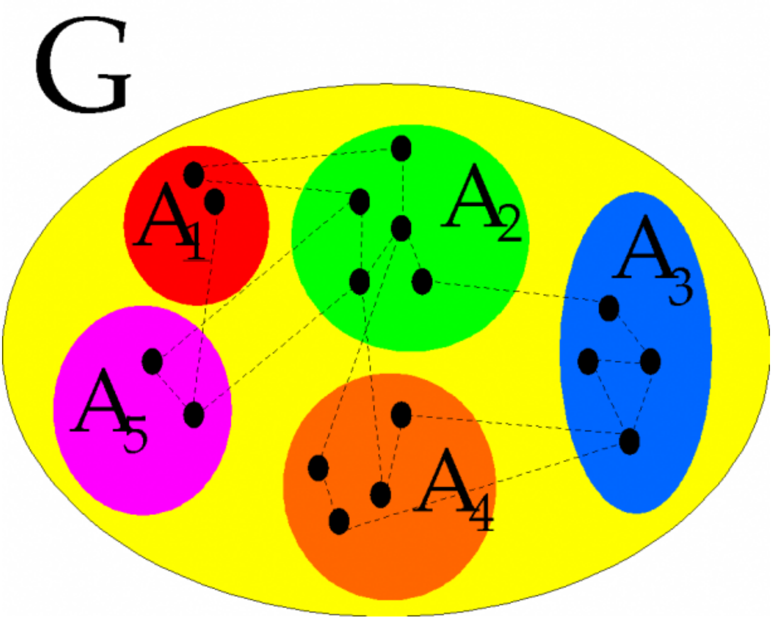
Large networks





Endre Szeméredi regularity theorem

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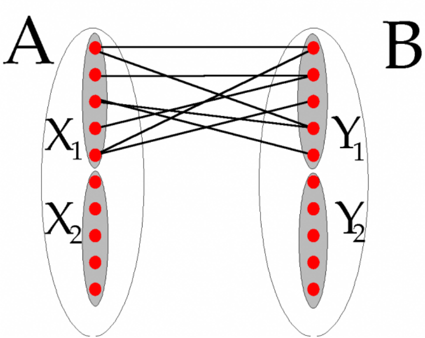
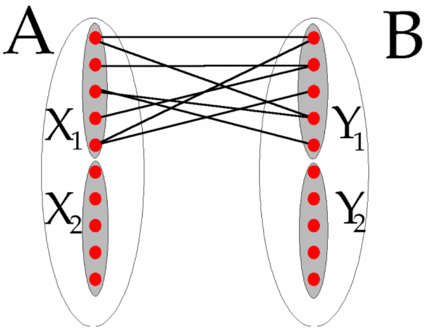
A_1
A_2
A_3
A_4
A_5



- density : $d(X_1, X_2) = \frac{e(X_1,X_2)}{|X_1||X_2|}$
- A, B is ϵ-regular if for every $X_1 \subset A$ with $|X_1| > \epsilon|A|$ and every $Y_1 \subset B$ with $|Y_1| > \epsilon|B|$, we have

$$|d(X_1, X_2) - d(A, B)| < \epsilon$$



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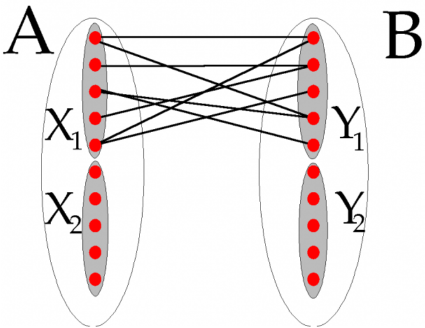

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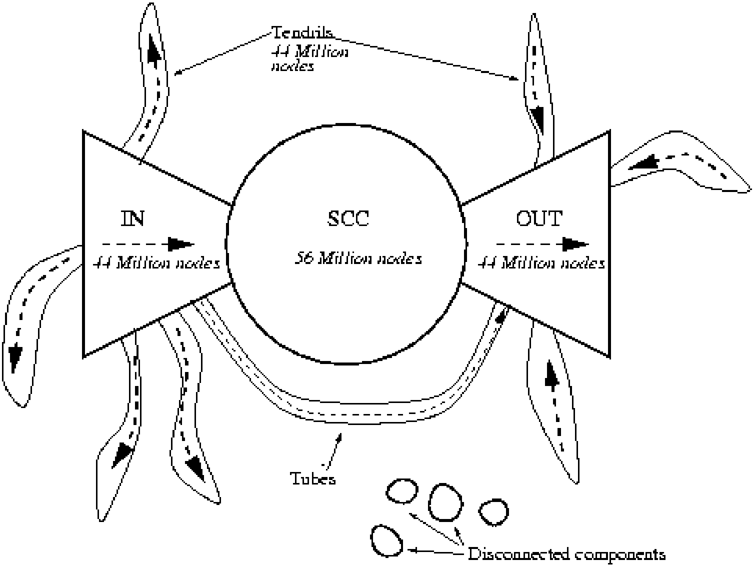
Endre Szeméredi regularity theorem

For every $\epsilon > 0$, there is m, M such that every finite graph can be partitioned in n parts A_i in such a way that

- $m \leq n \leq M$
- All A_i have approximately the same size : $(1-\epsilon)|A_i| \leq |A_j|(1+\epsilon)|A_i|$.
- Amon the n^2 pairs (A_i, A_j) at least $(1-\epsilon)n^2$ are ϵ-regular.



Tendrils
44 Million nodes
IN
44 Million nodes
SCC
56 Million nodes
OUT
44 Million nodes
Tubes
Disconnected components