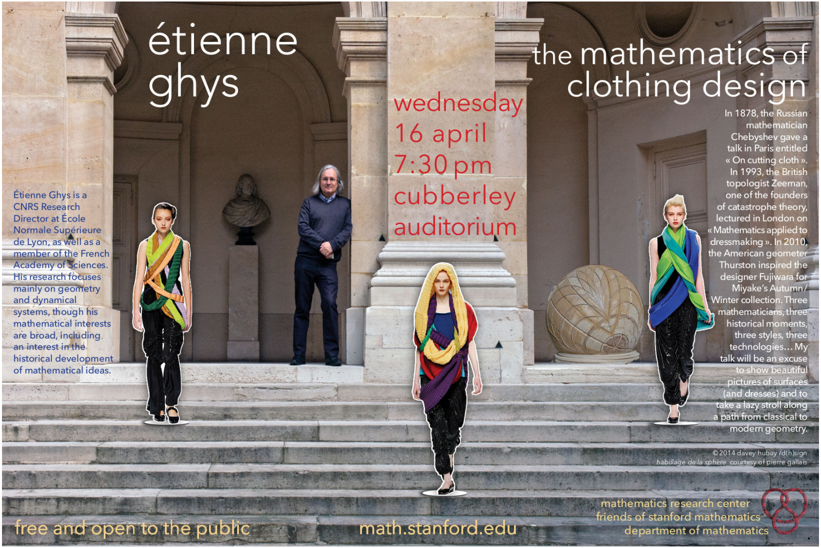
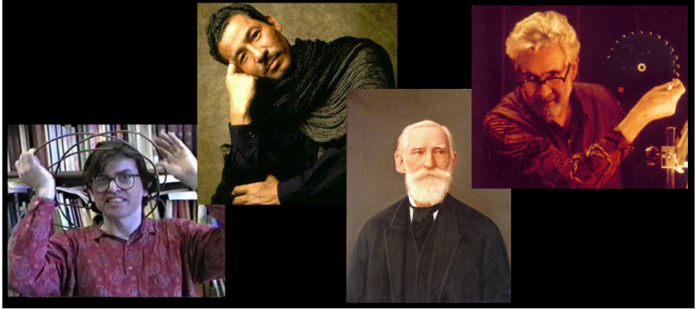
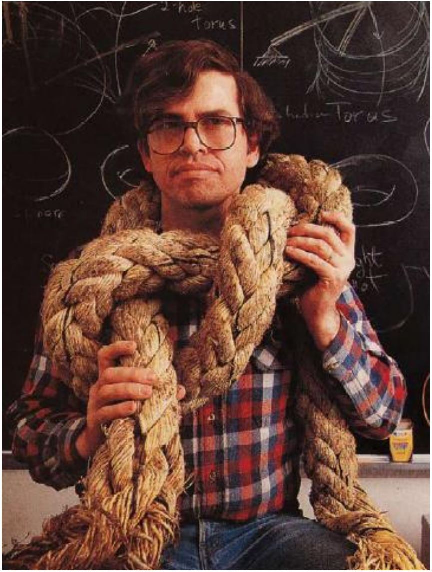


Davey Hubay / d(h)sign



William Thurston (1946-2012)



web.math.princeton.edu

Chapter I.5
Pleated Surfaces

I.5.1. Introduction

We now discuss pleated surfaces, which are a basic tool in Thurston's analysis of hyperbolic structures on 3-manifolds. See Section 8.8 of Thurston (1979); there, pleated surfaces are called uncrumpled surfaces.

Recall from definition under Section I.1.3.3 (*Isometric map*) that an isometric map takes rectifiable paths to rectifiable paths of the same length.

I.5.1.1. Definition. A map $f: M \to N$ from a manifold M to a second manifold N is said to be *homotopically incompressible* if the induced map $f_*: \pi_1(S) \to \pi_1(M)$ is injective.

I.5.1.2. Definition. *A pleated surface* in a hyperbolic 3-manifold M is a complete hyperbolic surface S together with an isometric map $f: S \to M$ such that every point $s \in S$ is in the interior of some geodesic arc which is mapped by f to a geodesic arc in M. We shall also require that f be homotopically incompressible.

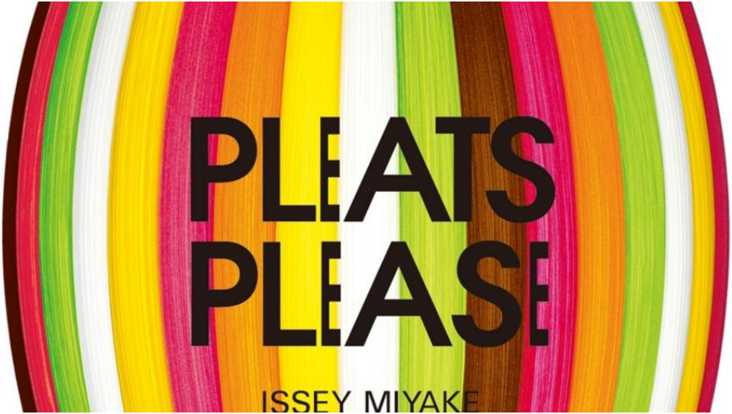
Note that this definition implies that a pleated surface f maps cusps to cusps since horocyclic loops on S are arbitrarily short and f is isometric and homotopically incompressible.

I.5.1.3. Definition. If (S, f) is a pleated surface, then we define its *pleating locus* to be those points of S contained in the interior of one and only one geodesic arc which is mapped by f to a geodesic arc.

Issey Miyake (1938-)

madmoizelle.com

Issey Miyake (1938-)



Issey Miyake (1938-)

Issey Miyake (1938-)

Issey Miyake (1938-)

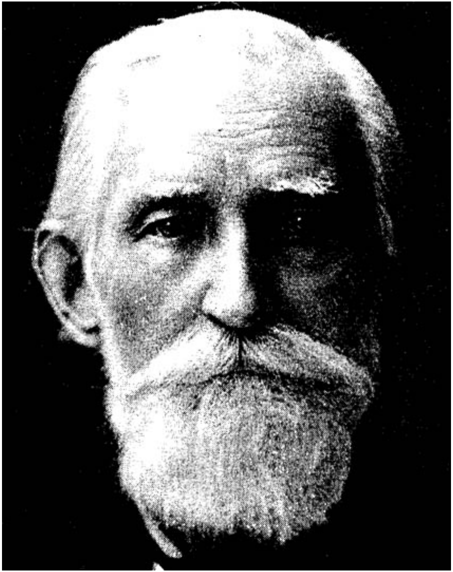
Issey Miyake (1938-)

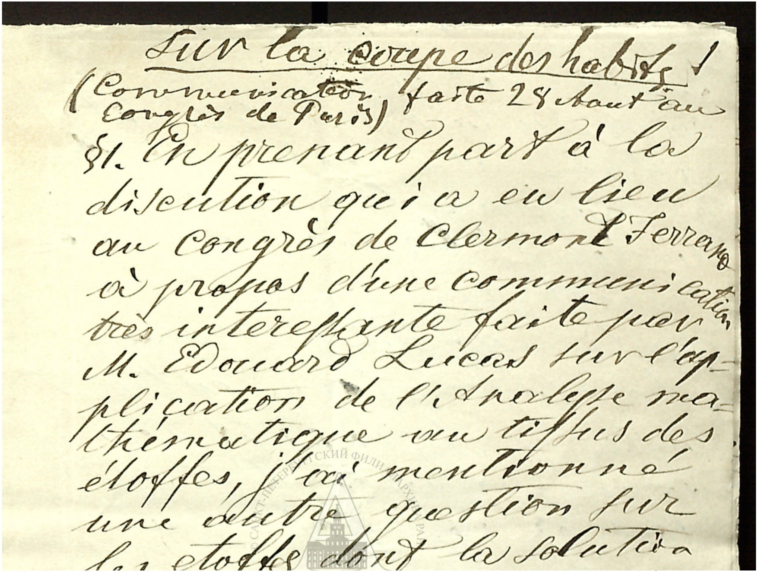
Thurston on Miyake Fashion Show, Paris 2010

with Dai Fujiwara, artistic director Miyake

Miyake Fashion Show Women's Ready to Wear, Paris 2010 -

Pafnuti Chebyshev (1821-1894)





Sur la coupe des habits

(Communication faite 28 Aout au Congrès de Paris)

§1. En prenant part à la discution qui a eu lieu au Congrès de Clermont Ferrand à propos d'une communication très interessante faite par M. Edouard Lucas sur l'application de l'Analyse mathématique au tissus des étoffes, j'ai mentionné une autre question sur les etoffes dont la solution

Sir Erik Christopher Zeeman (1925-)

Sir Erik Christopher Zeeman (1925-)

warwick.ac.uk

Sir Erik Christopher Zeeman (1925-)

Costume, Vol. 28, 97-102 (1994)

Mathematics Applied to Dressmaking

A Lecture Given at Gresham College, 9 March 1993

By PROFESSOR SIR CHRISTOPHER ZEEMAN, FRS

Gresham Professor of Geometry

DRESSMAKING can raise interesting questions in both geometry and topology. My own involvement began in Bangkok, where I once bought a dress-length of some rather beautiful Thai silk. Unfortunately when I got home all the dressmakers claimed it wasn't long enough to make a dress. It became clear that I had to either abandon the project or make the thing myself.

Paper clothing

Fashion paper

Issey Miyake fall/winter 2011-12 show

Fashion paper

Short Coat Issey Miyake 1985 Costumes ; Handmade mulberry fiber paper

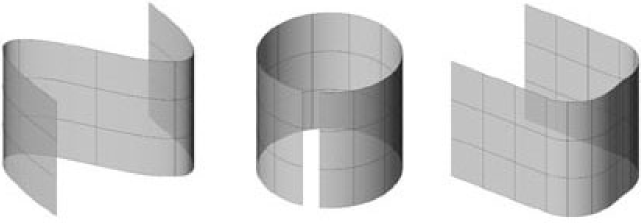
Fashion paper

Issey Myake's "8 Dresses In Pleats Paper"

Leonhard Euler (1707-1783)

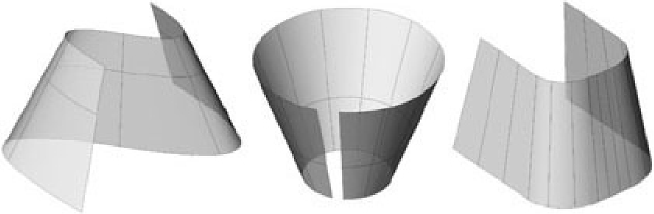
Which surfaces can be covered with paper,
allowing bending but not stretching ?

Which surfaces can be clothed with paper ?



Cylinders

Which surfaces can be clothed with paper ?



Cones

Paper Sculpture Fashion by Zaharova and Plotnikov

Paper Sculpture Fashion by Zaharova and Plotnikov

Paper Sculpture Fashion by Zaharova and Plotnikov

Euler : “de solidis etc.” Saint Peterburg, 1770

DE
SOLIDIS QVORVM
SVPERFICIEM IN PLANVM EXPLICARE LICET.

Auctore

L. EVLERO.

1.

Notiſſima eſt proprietas cylindri et coni, qua eorum ſuperficiem in planum explicare licet atque adeo haec proprietas ad omnia corpora cylindrica et conica extenditur, quorum baſes figuram habeant quamcunque; contra vero ſphaera hac proprietate deſtituitur, quum eius ſuperficies nullo modo in planum explicari neque ſuperficie plana obduci queat; ex quo naſcitur quaeſtio aeque curioſa ac notatu digna, vtrum praeter conos et cylindros alia quoque corporum genera exiſtant, quorum ſuperficiem itidem in planum explicare liceat nec ne? quam ob rem in hac diſſertatione ſequens conſiderare conſtitui Problema:

Inuenire aequationem generalem pro omnibus ſolidis, quorum ſuperficiem in planum explicare licet, cuius ſolutionem variis modis ſum agreſſurus.

Definition : *A surface is developable if it can be covered with paper allowing bending but not stretching.*

Theorem (Euler 1770) : *A developable surface is necessarily* **ruled**. *Through every point there is a straight line that lies on the surface.*

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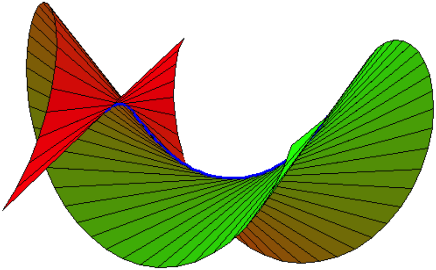
Theorem (Euler 1770) : *A developable surface is necessarily* **ruled**. *Through every point there is a straight line that lies on the surface.*

Most ruled surfaces are not developable.

Theorem (Euler 1770) : *A ruled surface is developable if and only if it is either a cone or a cylinder or is formed by the union of the tangents to some space curve.*



wikipedia.org/wiki/File :Helixtgtdev.png

Developable surface

Antoine-Pevsner-Developable-Surface-1938

Developable surface

http ://www.omkrets.se

Lebesgue's handkerchief : counterexample to Euler's theorem ?

Theorem (Lebesgue 1899) : Some surfaces are developable and not ruled !

NOT smooth. Continuous surfaces with no tangent planes.

Lebesgue's handkerchief : counterexample to Euler's theorem ?

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Issey Miyake *Minaret dress* : Developable surface ? Ruled ?

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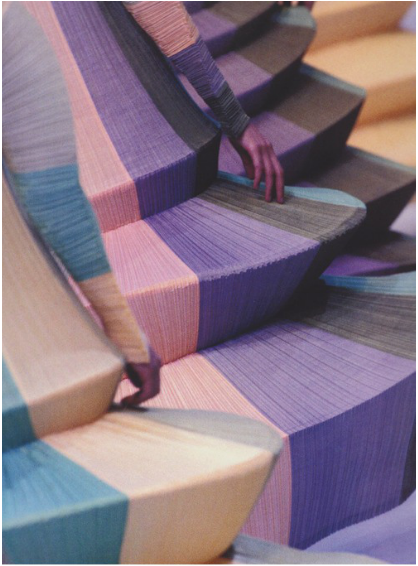
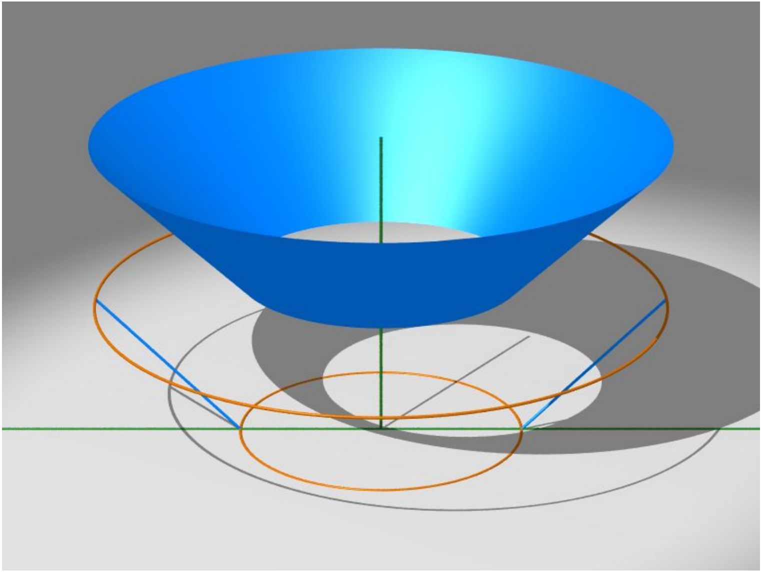


Issey Miyake *Minaret dress* : Developable surface ? Ruled ? -

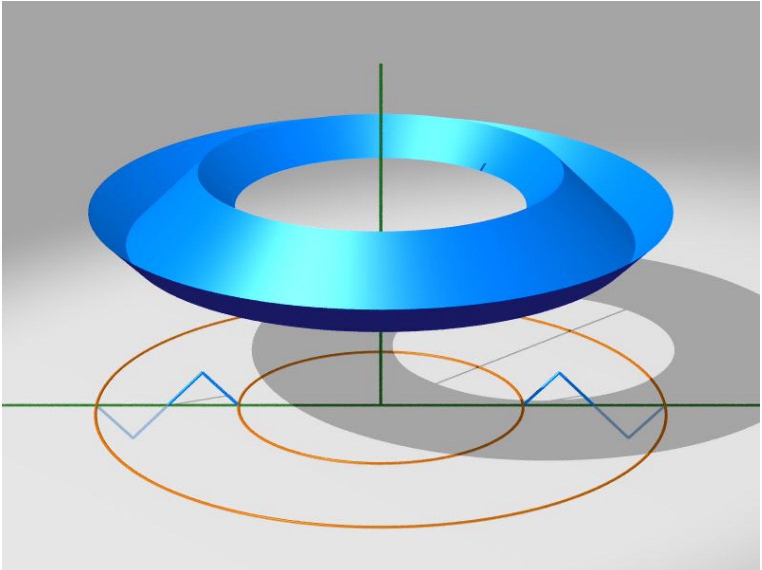
A developable surface, Issey Miyake Milan 1995

-

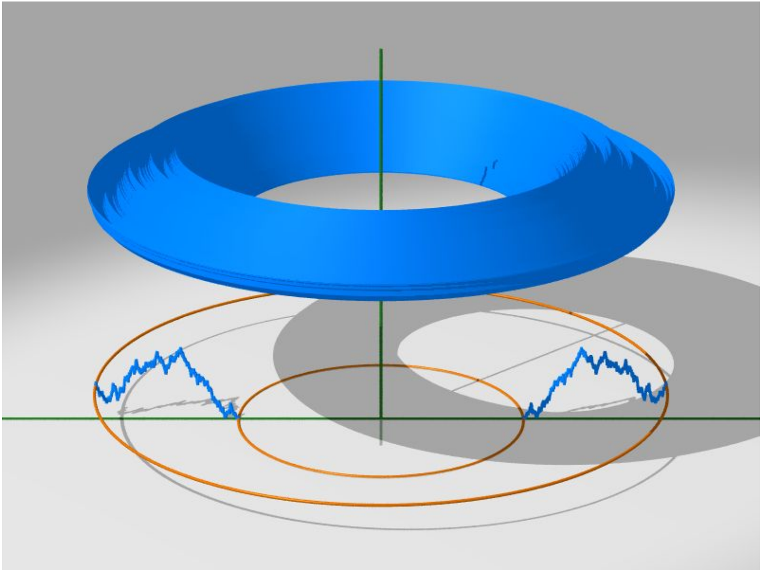
Lebesgue



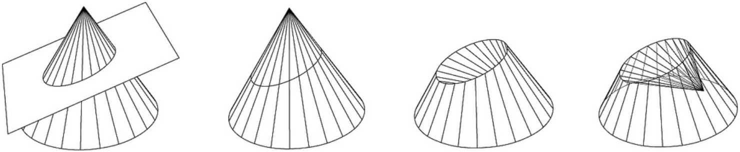
Lebesgue



Lebesgue



Pleated surface

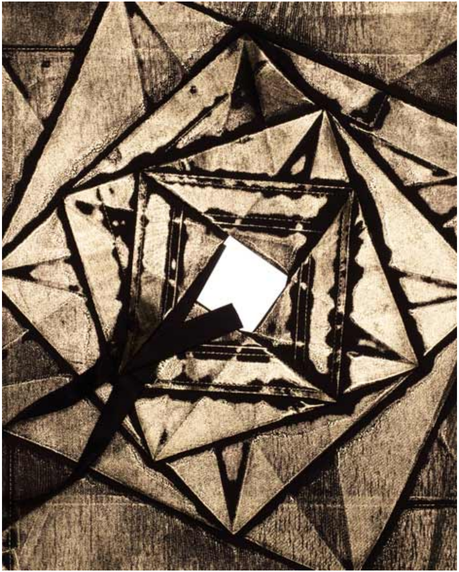


Flickr Tachi

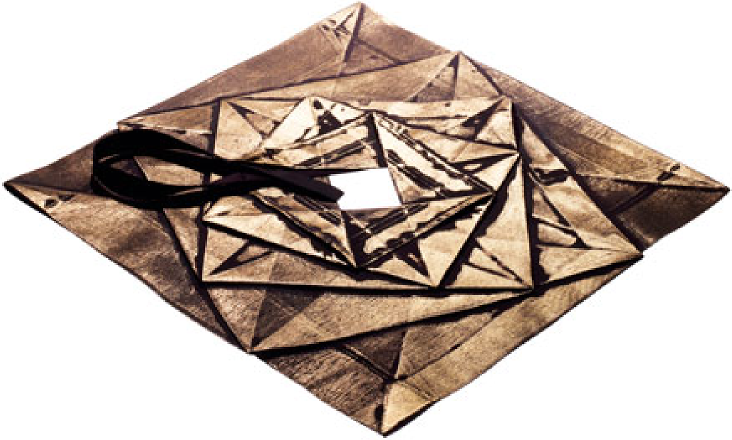
Pleated surface

Huffman

Issey Miyake, developable dress, origami



Issey Miyake, developable dress, origami



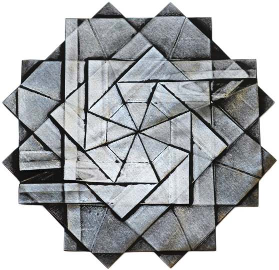
Issey Miyake, developable dress, origami

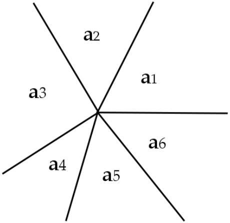
Issey Miyake, developable dress, origami

Issey Miyake, developable dress, origami

Issey Miyake, developable dress, origami

Theorem (Kawasaki, 1989) : *Let us fold a paper. Radiating from any vertex, there is an even number of creases, and the alternating sum of the angles is 0.*





$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0$$

A developable surface, Issey Miyake Fall Winter 2014-2015 -

designer Yoshiyuki Miyamae

Some surfaces are developable, smooth AND not ruled !

Theorem (Nash 1954) : *There exists a surface which*

- *is smooth, i.e. has a tangent plane everywhere,*
- *is developable : can be clothed with paper,*
- *is not ruled : it contains no segment.*

They are smooth, with no tangent planes but they do not have a second derivative. Euler theorem requires two derivatives.

Some surfaces are developable, smooth AND not ruled !

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Flat, smooth and not ruled

Borrelli, Jabrane, Lazarus, Thibert (2012)

Chebyshev

Cloth, not paper !

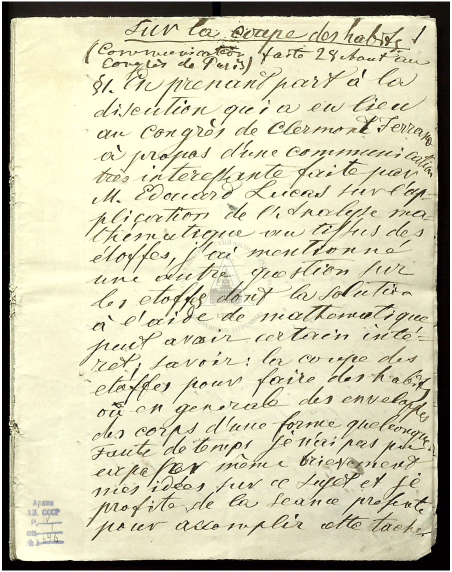
Sur la coupe des vêtements.

(Association française pour l'avancement des sciences. 7 session. Paris. Séance du 28 août 1878).

Après avoir indiqué que l'idée de cette étude lui est venue lors de la communication faite, il y a deux ans, au Congrès de Clermont-Ferrand, par M. Edouard Lucas, sur la géometrie du tissage des étoffes à fils rectilignes, M. Tchébichef pose les principes généraux pour déterminer les courbes suivant lesquelles on doit couper les differents morceaux d'une étoffe, pour en faire une gaine bien ajustée, servant à envelopper un corps de forme quelconque.

En prenant pour point de départ ce principe d'observation que dans la déformation d'un tissu on ne doit considérer d'abord, dans une première approximation, que l'altération des angles respectifs formés par les fils de chaîne et les fils de trame, sans tenir compte de l'allongement des fils, il donne les formules qui permettent de déterminer les contours imposés à deux, trois ou quatre morceaux d'étoffe pour recouvrir la surface d'une sphère, avec la meilleure approximation désirable. M. Tchébichef présente à la section une balle de caoutchouc recouverte d'une étoffe dont les deux morceaux ont été coupés suivant ses indications; il fait observer que le problème différerait essentiellement si l'on remplaçait l'étoffe par une peau. D'ailleurs les formules proposées par M. Tchébichef donnent aussi la méthode à suivre pour la juxtaposition des pièces par la couture.

Conformément à la volonté de Tchebychef, l'étude «Sur la coupe des habits» trouvée dans ses papiers ne doit pas être imprimée, car le manuscrit ne porte pas l'inscription: «*imprimer*».



Sur la coupe des habits.
(Communication faite 28 Août au Congrès de Paris)

§1. En prenant part à la discution qui a eu lieu au Congrès de Clermont Ferrand à propos d'une communication très interessante faite par M. Edouard Lucas sur l'application de l'Analyse mathématique au tissus des étoffes, j'ai mentionné une autre question sur les étoffes dont la solution à l'aide de mathematique peut avoir certain intérêt, savoir: la coupe des étoffes pour faire des habits où en generale des enveloppes des corps d'une forme quelconque. Faute de temps je n'ai pas pu exposer même brièvement mes idées sur ce sujet et je profite de la seance présente pour accomplir cette tache.

6

On y parvient très aisément d'après l'équation de la courbe de la moindre distance, donnée par cette formule du calcul de variation

$$\delta\int dS = 0,$$

qui d'après (1) se réduit à celle-ci

$$\delta\int\sqrt{dx^2+dy^2+2dx\,dy\cos\varphi} = 0$$

et ~~δ∫~~ d'où l'on tire l'équation,

$$(3)\quad \sin^2\varphi\cdot\frac{d^2y}{dx^2}+\frac{d\cos\varphi}{dx}\left(1+\cos\varphi\frac{dy}{dx}\right)-\frac{d\cos\varphi}{dy}\left(\cos\varphi+\frac{dy}{dx}\right)\frac{dy}{dx} = 0$$

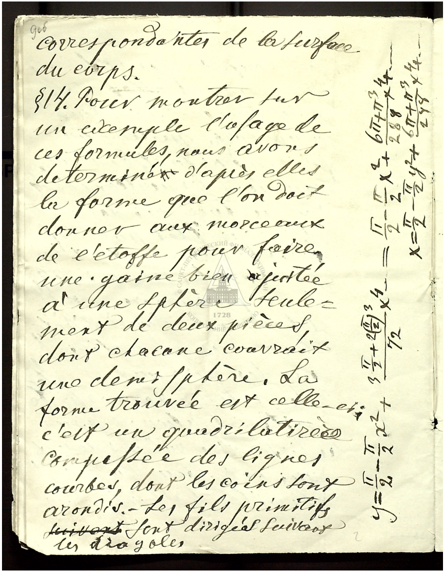
§10. En remarquant que l'axe de x, dont l'équation est

$$y = 0,$$

profite, comme nous l'avons dit, l'une des courbes de la plus courte distance, nous trouvons en y appliquant l'équation précédente

$$\frac{d\cos\varphi}{dx} = 0,$$

Sur la coupe des habits

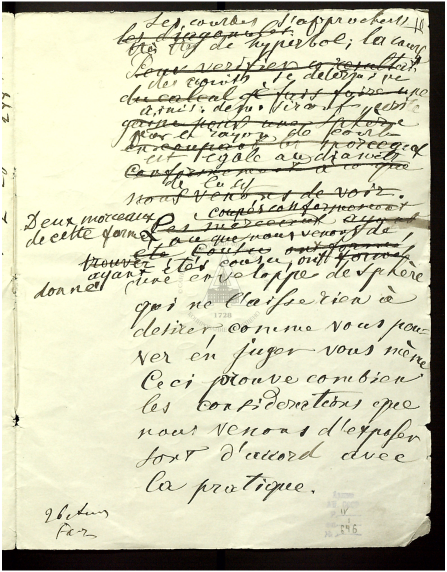


correspondantes de la surface du corps.

§14. Pour montrer sur un exemple l'usage de ces formules, nous avons déterminé d'après elles la forme que l'on doit donner aux morceaux de l'étoffe pour faire une gaine bien ajustée à une sphère seulement de deux pièces, dont chacune couvrait une demi sphère. La forme trouvée est celle-ci: c'est un quadrilatère composée des lignes courbes, dont les coins sont arondis. – Les fils primitifs ~~suivent~~ sont dirigés suivant les diagoles

$$y=\frac{\pi}{2}-\frac{\pi}{2}x^2+\frac{3\frac{\pi}{2}+2\left(\frac{\pi}{2}\right)^3}{72}x^4 \ldots = \frac{\pi}{2}-\frac{\pi}{2}x^2+\frac{6\pi+\pi^3}{288}x^4 \ldots$$

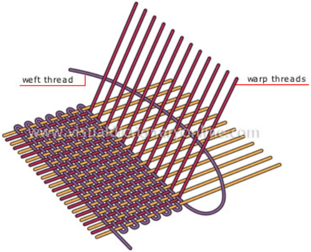
$$x=\frac{\pi}{2}-\frac{\pi}{2}y^2+\frac{6\pi+\pi^3}{288}x^4 \ldots$$

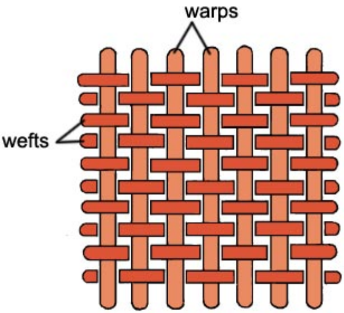

Les courbes s'approchent
~~les diagonales.~~
très très de hyperbol; la courbe
~~Pour verifier ce resultat~~
des equations je determine
~~du calcul~~ [illegible] ~~fais faire une~~
[illegible]
~~[illegible] pour une sphère~~
par le rayon de [illegible]
~~en coupant les morceaux~~
est egale au diametre
~~conformement à ce que~~
de [illegible]
~~nous venons de voir.~~

Deux morceaux de cette forme ~~Les morceaux ayant~~ coupés conformement à ce que nous venons de ~~trouver~~ ~~été coutes ont formé~~ ayant étés coupés, ont donné une enveloppe de sphère qui ne laisse rien à desirer comme vous pouvez en juger vous même. Ceci prouve combien les considerations que nous venons d'exposer sont d'accord avec la pratique.

26 Avr [illegible]

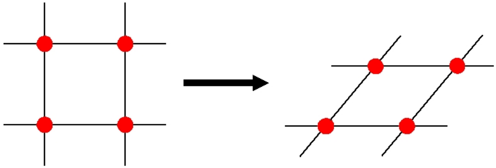
Sur la coupe des habits

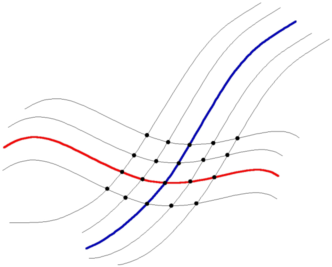




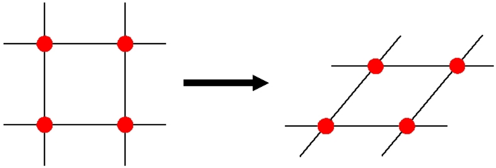
fab.cba.mit.edu

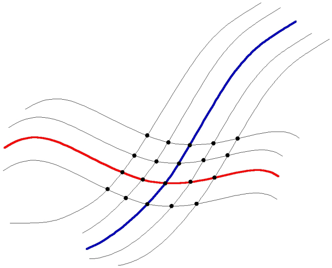
Cloth and not paper





Cloth and not paper

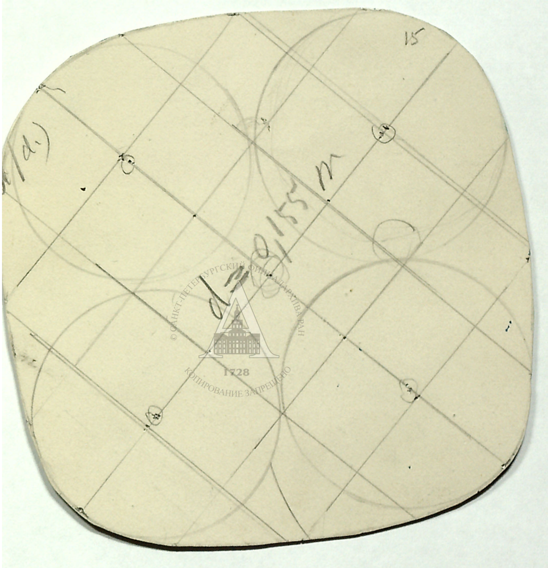




Sur la coupe des habits

Theorem (Chebyshev 1878) *It is possible to clothe half the sphere. There exists a template that can be deposited on an hemisphere in such a way that the infinitesimal warp-weft squares are mapped to rhombuses.*

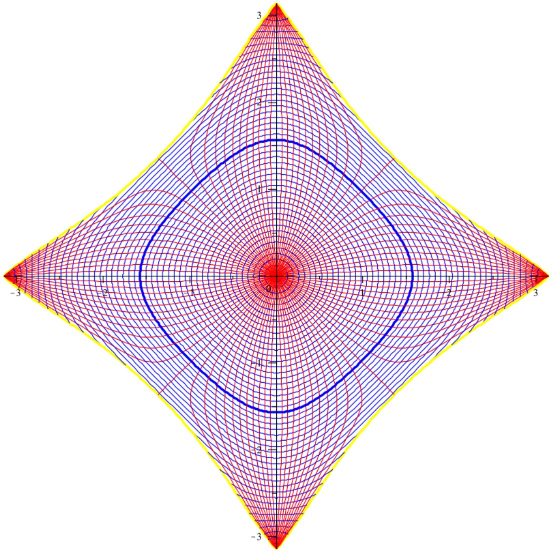
A template for the hemisphere



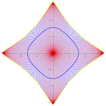
Sur la coupe des habits

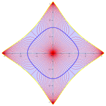
Theorem (2011) *It is possible to clothe* **the full** *sphere. There exists a template that can be deposited on a sphere in such a way that the infinitesimal warp-weft squares are mapped to rhombuses.*

A template for the sphere



A template for the sphere

There is a map Φ from  to the sphere  such that :

- Φ preserves length along warp and weft, ie $\frac{\partial \Phi}{\partial x}$ and $\frac{\partial \Phi}{\partial y}$ have length 1.
- The image of Φ covers the sphere exactly once (away from the seams, on the boundary of ).

Clothing a ball

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

Pierre Gallais

3 D printing of Chebyshev

3D printed dress by Michael Schmidt and Francis Bitonti

Dita Von Teese

3D printed dress by Michael Schmidt and Francis Bitonti

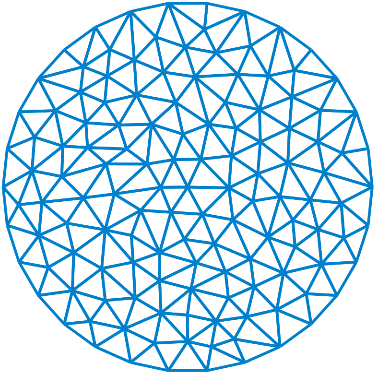
Dita Von Teese

Dita Von Teese

Nervous.com

Flexibility

A triangulated disk is flexible.

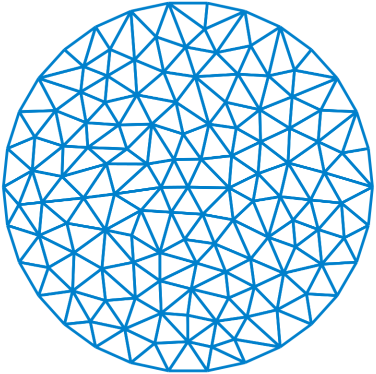


$$\sharp \text{Degrees of freedom} = 3\sharp\text{Vertices} - \sharp\text{Edges} - 6$$

$$= \sharp\text{Exterior Edges} - 3$$

Flexibility

A triangulated disk is flexible.



$$\sharp\text{Degrees of freedom} = 3\sharp\text{Vertices} - \sharp\text{Edges} - 6$$

$$= \sharp\text{Exterior Edges} - 3$$

Rigidity

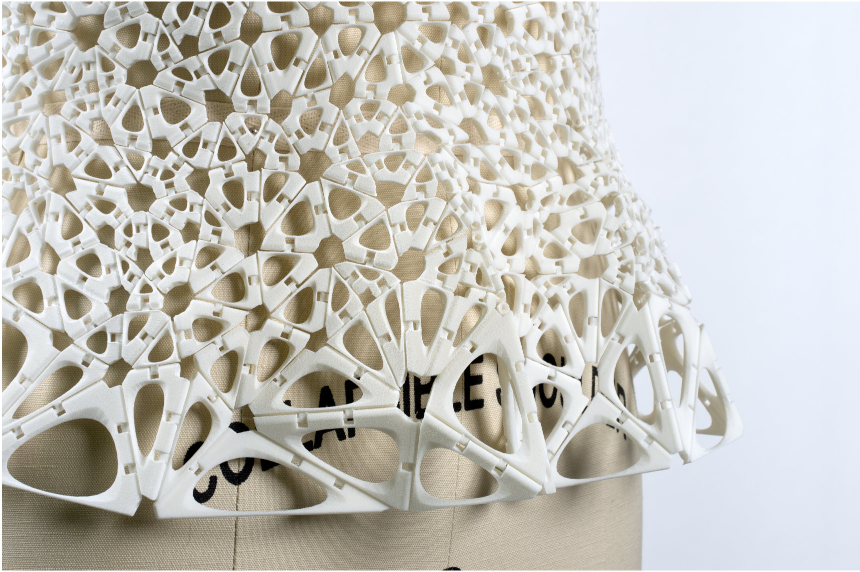
A triangulated sphere is (usually) rigid.

$$\sharp\text{Degrees of freedom} = 3\sharp\text{Vertices} - \sharp\text{Edges} - 6 = 0$$

Flexible bags, Miyake collection, Fall Winter 2012

Nervous.com

Nervous.com



Zeeman and catastrophe theory

Christopher Zeeman

Christopher Zeeman

Cusps and folds

Christopher Zeeman

Costume, Vol. 28, 97-102 (1994)

Mathematics Applied to Dressmaking

A Lecture Given at Gresham College, 9 March 1993

By PROFESSOR SIR CHRISTOPHER ZEEMAN, FRS

Gresham Professor of Geometry

DRESSMAKING can raise interesting questions in both geometry and topology. My own involvement began in Bangkok, where I once bought a dress-length of some rather beautiful Thai silk. Unfortunately when I got home all the dressmakers claimed it wasn't long enough to make a dress. It became clear that I had to either abandon the project or make the thing myself.

Christopher Zeeman

Now, I had never made a dress before, nor any garment for that matter, and so I thought it would be amusing to try designing it from scratch. Fools rush in where angels fear to tread. In my innocence I chose a simple sleeveless summer dress with a princess line; or in layman's language it just consisted of two panels, one at the front and one at the back, sewn together at the sides. What I had not realized was that, the simpler the dress, the more accurately it has to fit. As a precaution I first tried a mock-up made out of an old sheet. And a good thing, too, because the result was hopeless: when she tried it on it hardly fitted anywhere. I slowly began to realize that I did not yet understand the basic mathematical problem of how to fit a flexible flat surface round a curved surface. So back to the drawing board to do a little differential geometry.

Christopher Zeeman

Cutting 'on the cross' or 'on the bias' merely means arranging things so that the fibres of the warp and weft go diagonally relative to the body instead of going horizontally and vertically. Therefore, if the dress is pulled tight horizontally and vertically, then the material will be stretched diagonally relative to the fibres. Now, it is a characteristic property of woven material that it cannot be stretched *along* the fibres but it can be stretched *diagonal* to them. And if it is stretched along both diagonals at the same time then it will form a surface of negative curvature like a quadric, such as $xy = z$ (see Fig. 1). In a quadric the fibres are in fact straight lines (as in a plane), but they are no longer parallel to one another, and consequently they must have been pulled slightly apart round the perimeter (or pushed slightly closer together in the middle, or both). Incidentally, it is a classical theorem of projective geometry that any non-planar surface containing two intersecting families of lines is a quadric.

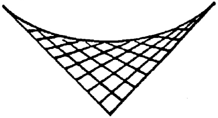


FIG. 1.

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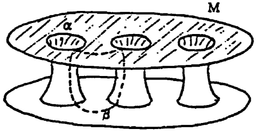


FIG. 7.

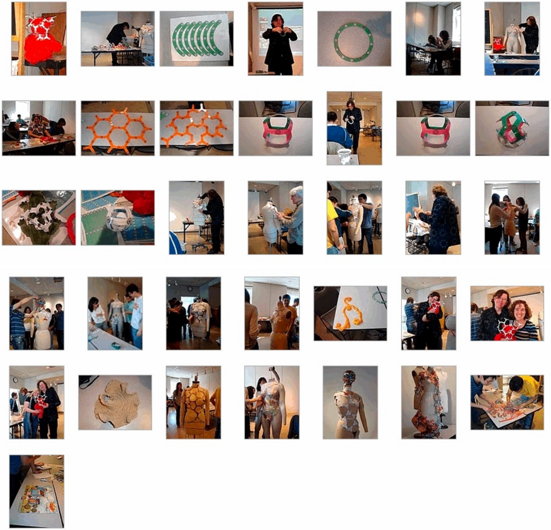
movement the surface occupies exactly the same position that it did at the beginning, except that individual points on the surface have been moved around. The surprising feature is the reversal of the sides. A similar isotopy is illustrated in Fig. 6, but that is easier to understand because there the surface is only of genus 1. In Fig. 7 if we identify the top of M with the dress, the bottom of M with the lining, and the three holes with the neck and armholes, then the theorem seems to imply that it is topologically possible to turn the dress inside out, after all. It was only after some thought that I realized the isotopy would also move the curve α into the curve β, because α can be spanned by a disk on only one side of the surface, while β can be spanned on the other. Therefore if the isotopy were applied to the dress, then the identity of the neck and arm holes would be lost. Hence the correct topological obstruction is: there is no side-reversing isotopy of the *pair* (M, α).

NOTE *by Lady Zeeman*

The length of cloth brought back from Bangkok, and declared by all to be 'insufficient for a dress', was ivory-coloured Thai silk, with two bands of delicate colour, one dove-grey and the other *café au lait*, woven into either end. From this, Sir Christopher designed and made a princess-line dress, doing all the cutting, machining and hand-finishing himself. It had a scoop neckline, no sleeves, and fell to just below the knee, with an eight-inch zip hidden in the side seam. It was lined throughout, and skimmed the figure in a manner both comfortable and elegant. It was a great pleasure to wear as well as being very pleasing to the eye. Unfortunately, the figure outgrew the dress in time, and it was given away. But a long evening skirt of five panels, in chestnut-coloured velvet with cream flowers, also designed and made by Sir Christopher for his wife, is still in frequent use.

Back to paper, with W. Thurston

William Thurston

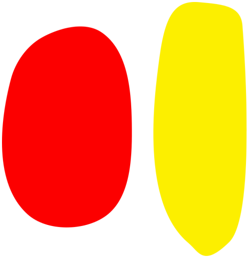


William Thurston

William Thurston

hyperbolic-crochet

Theorem (Pogorelov 1970) : *Start with two convex domains with the same perimeter. Glue them along their boundaries (respecting the arc lengths). One gets a convex body in space which is rigid.*



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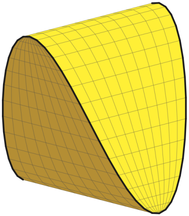
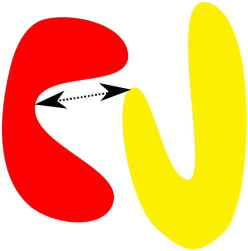


Theorem (Pogorelov 1970) : *The same is true if the two domains are not convex but if the sum of the two curvatures at points which are identified is positive.*



Playing with Surfaces: Spheres, Monkey Pants, and Zippergons

Kelly Delp
Mathematics Department
Buffalo State College
kelly.delp@gmail.com

Bill Thurston
Mathematics Department
Cornell University
wpt@math.cornell.edu

Abstract

We describe a process, inspired by clothing design, of smoothing an octahedron to form a round sphere. This process can be adapted to construct many different surfaces out of paper and craft foam.

Introduction

(a) *Paper and tape model*

(b) *Pattern*

Figure 1: *Octahedral Sphere*

Kelly Delp, William Thurston, Bridges Coimbra Conference Proceedings (2011), 1-8.

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Flicker David Swart

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Bill Wingell for The New York Times