Groups acting on the circle: a selection of open problems.

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Diff_+^k(S^1) as a group

**Theorem (Mather, Thurston)**

\( \text{Diff}_+^k(S^1) \) is a simple group, at least if \( k \neq 2 \).

**Question**

Is \( \text{Diff}_+^2(S^1) \) a simple group?

**Remark (Mather)**

\( \text{Diff}_+^{1+bv}(S^1) \) is not a simple group.
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*What is the cohomology of $\text{Diff}^+_k(S^1)$ as a discrete group?*

**Theorem (Mather)**

The cohomology of $\text{Diff}^0_+(S^1)$ is generated by the Euler class.

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*If the rotation number of an element $f$ of $\text{Diff}^2_+(S^1)$ is irrational, then $f$ is conjugate to a rotation by a homeomorphism. In particular, there is a unique invariant probability measure.***

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If the rotation number of an element $f$ of $\text{Diff}^\infty_+(S^1)$ satisfies a diophantine condition, then $f$ is smoothly conjugate to a rotation. In particular, the invariant probability is smooth.

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*There is a unique stationary probability measure.*

Assume furthermore that the generators are close enough to the identity. Then

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