Groups acting on the circle: a selection of open problems.

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$\mathrm{Diff}_+^k(\mathbf{S}^1)$ as a group

Theorem (Mather, Thurston)

$\mathrm{Diff}_+^k(\mathbf{S}^1)$ *is a simple group, at least if* $k \neq 2$.

Question

Is $\mathrm{Diff}_+^2(\mathbf{S}^1)$ *a simple group?*

Remark (Mather)

$\mathrm{Diff}_+^{1+bv}(\mathbf{S}^1)$ *is not a simple group.*

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$\mathrm{Diff}^k_+(\mathbf{S}^1)$ as a group : its cohomology

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What is the cohomology of $\mathrm{Diff}^k_+(\mathbf{S}^1)$ as a discrete group?

Theorem (Mather)

The cohomology of $\mathrm{Diff}^0_+(\mathbf{S}^1)$ is generated by the Euler class.

Theorem (Tsuboi)

The cohomology of $\mathrm{Diff}^1_+(\mathbf{S}^1)$ is generated by the Euler class.

Theorem (Thurston)

$\mathrm{H}_2(\mathrm{Diff}^k_+(\mathbf{S}^1), \mathbf{R})$ *surjects onto* $\mathbf{R} \oplus \mathbf{Z}$.

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$\mathrm{Diff}^k_+(\mathbf{S}^1)$ as a group: its continuous cohomology

Theorem (Gelfand-Fuchs)

The continuous cohomology of $\mathrm{Diff}^\infty_+(\mathbf{S}^1)$ is generated by the Euler class and the Godbillon-Vey class.

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What is the **continuous** *cohomology of $\mathrm{Diff}^{an}_+(\mathbf{S}^1)$ as a topological group?*

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Does the cube of the Euler class in $\mathrm{H}^6(\mathrm{Diff}^{an}_+(\mathbf{S}^1), \mathbf{R})$ vanish ?

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Compute the "formal cohomology" of the Virasoro Lie algebra.

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Theorem

The Godbillon-Vey class in $\mathrm{H}^2(\mathrm{Diff}^\infty_+(\mathbf{S}^1, \mathrm{R}))$ is a C^1 invariant.

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Consider an action of the fundamental group of a closed surface on the circle by C^∞ diffeomorphisms. Is it true that the action is Godbillon-Vey rigid if and only if the action is projective ?

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Subgroups of $\mathrm{Diff}^k_+(\mathbf{S}^1)$

Question

Let Γ be a finitely generated group. Is the space of representations of Γ in $\mathrm{Diff}^\infty_+(\mathbf{S}^1)$ locally connected ?

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Find all group actions on the circle which are structurally stable.

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Find all group actions on the circle which are amenable.

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Find "non trivial" examples of groups acting on the circle !

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If the rotation number of an element f of $\mathrm{Diff}^2_+(\mathbf{S}^1)$ is irrational, then f is conjugate to a rotation by a homeomorphism. In particular, there is a unique invariant probability measure.

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If f is a homeomorphism of the circle whose graph is analytic, does Denjoy theorem apply ?

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Theorem (Herman)

If the rotation number of an element f of $\mathrm{Diff}_+^\infty(\mathbf{S}^1)$ satisfies a diophantine condition, then f is smoothly conjugate to a rotation. In particular, the invariant probability is smooth..

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If the rotation number of an element f of $\mathrm{Diff}_+^2(\mathbf{S}^1)$ is irrational, is there a unique invariant **distribution***?*

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Dynamics of elements of $\mathrm{Diff}^k_+(\mathbf{S}^1)$

Let f be a diffeomorphism of the circle and $\epsilon > 0$. Glue the two boundaries of $\mathbf{S}^1 \times [0, \epsilon]$ using f. This produces an elliptic curve isomorphic to $\mathbf{C}/(\mathbf{Z} + \tau(\epsilon)i\mathbf{Z})$ with $\tau(\epsilon) \in \mathbf{R}/\mathbf{Z}$.

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Let Γ be a finitely generated group acting smoothly on the circle. Assume orbits are dense. Does that imply that the action is ergodic with respect to Lebesgue ?

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Let Γ be a finitely generated group acting smoothly on the circle. Assume that some orbits are not dense but that there is no finite orbit. This implies the existence of an invariant minimal Cantor set K. Does that imply that the Lebesgue measure of K is zero and that $\mathbf{S}^1 \setminus K$ consists of a finite number of orbits modulo the Γ-action ?

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Dynamics of subgroups of $\mathrm{Diff}^{\infty}_{+}(\mathbf{S}^1)$ generated by small elements

Let Γ be a group acting smoothly on the circle. Suppose that there is no finite orbit.

Theorem (Deroin-Kleptsyn-Navas)

There is a unique stationary probability measure.

Assume furthermore that the generators are close enough to the identity. Then

Theorem (Duminy)

All orbits are dense.

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Is the unique stationnary measure a smooth measure ?

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