Knots and Dynamics

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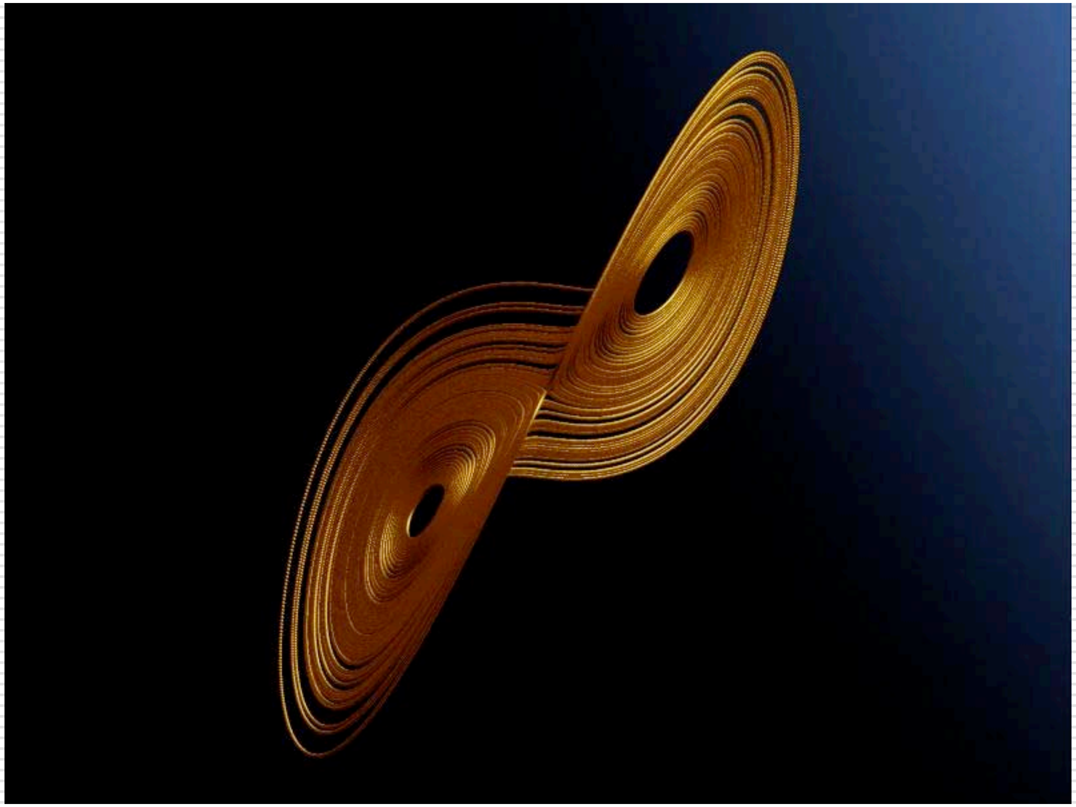
Lorenz equation (1963)

$$\frac{dx}{dt} = 10\,(y - x)$$

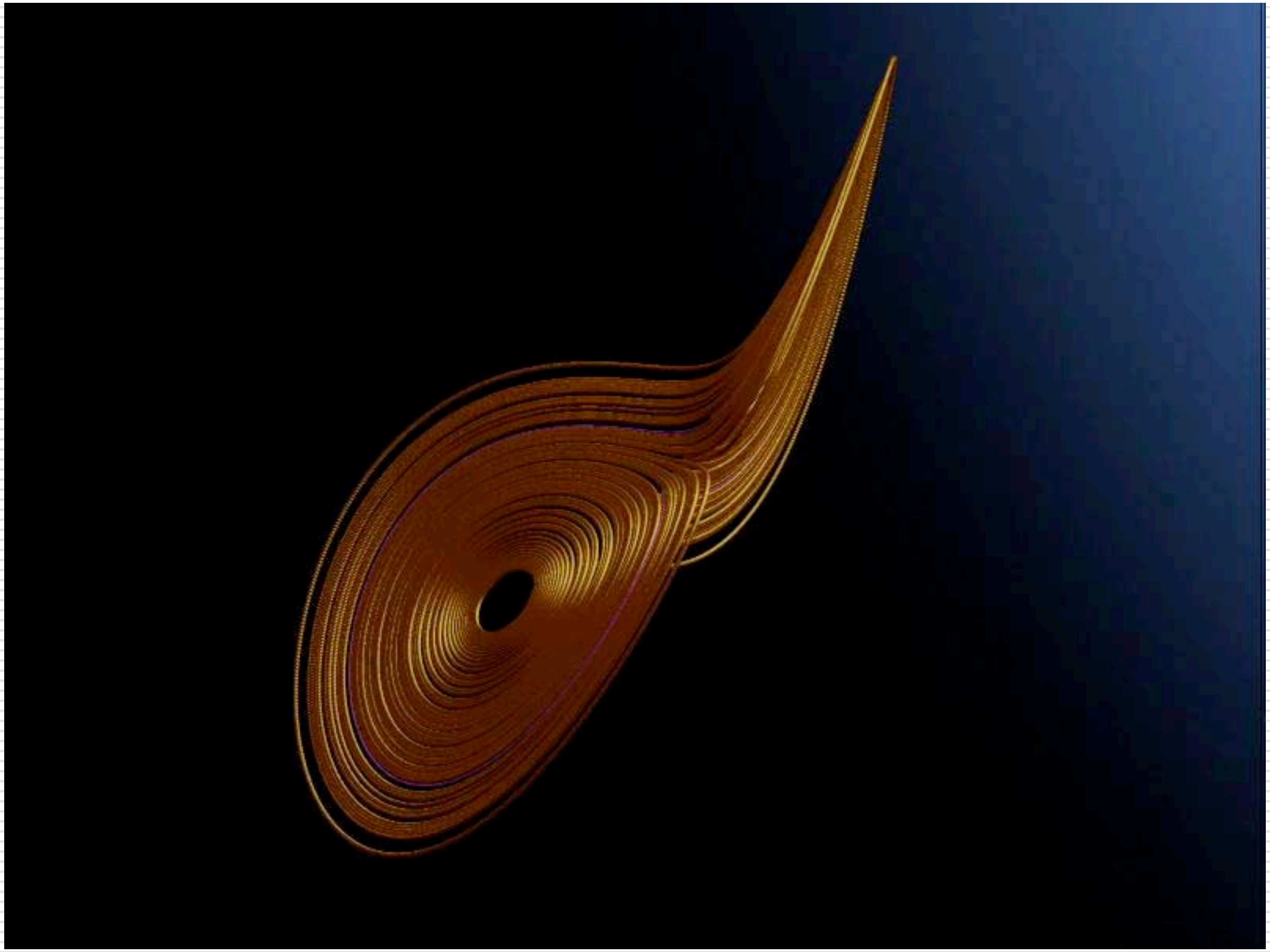
$$\frac{dy}{dt} = 28x - y - xz$$

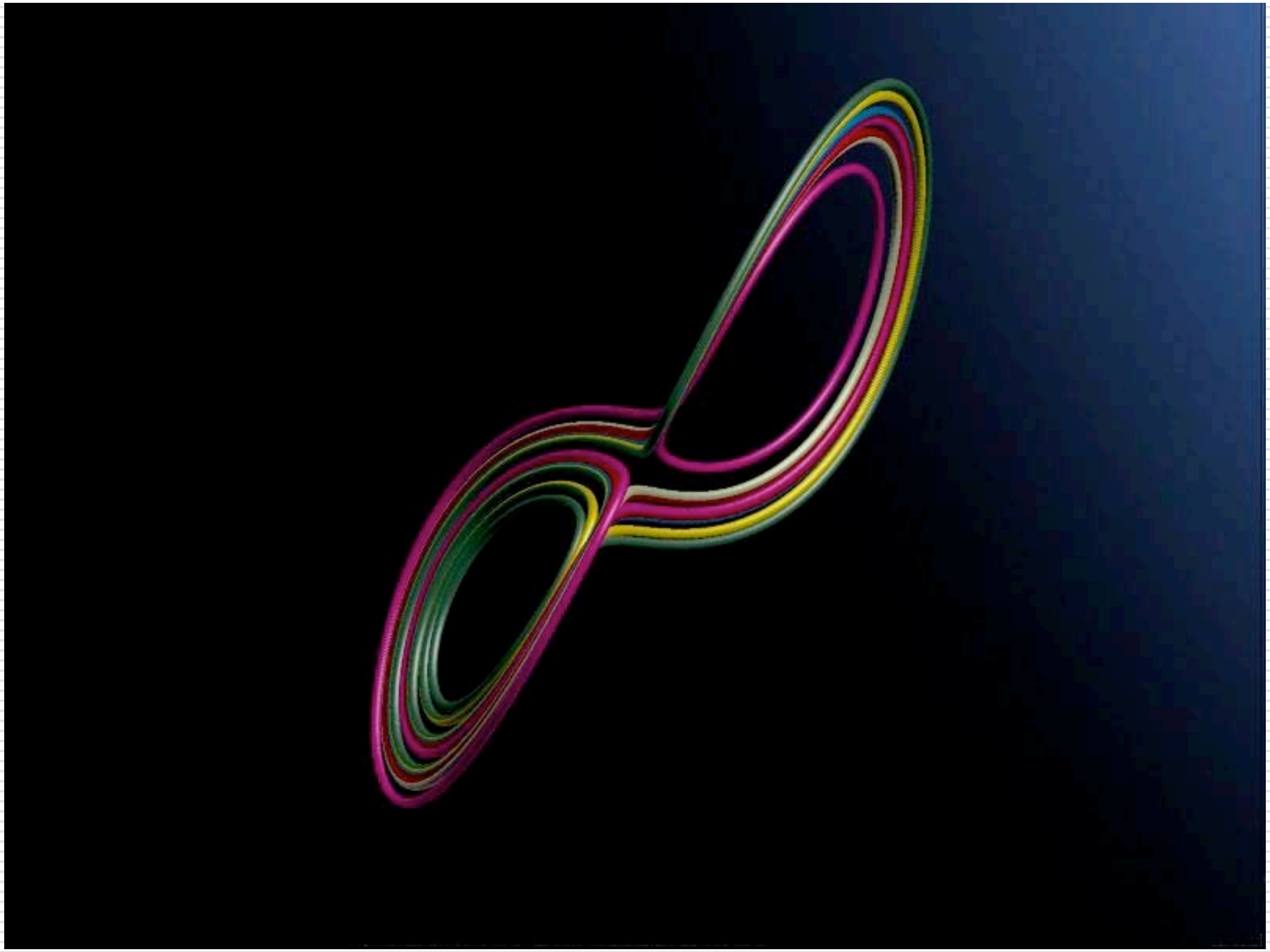
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$



Birman-Williams: Periodic orbits are ***knots***.

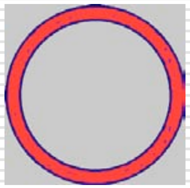
Topological description of the Lorenz attractor



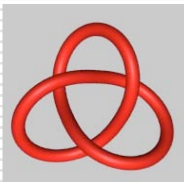


Birman-Williams: Lorenz knots and links are very peculiar

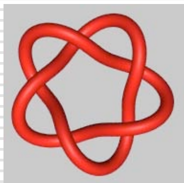
- Lorenz knots are prime
- Lorenz links are fibered
- non trivial Lorenz links have positive signature



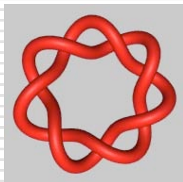
0_1 Unknot



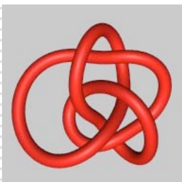
3_1 trefoil



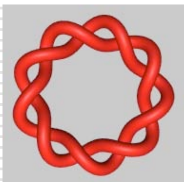
5_1Cinquefoil



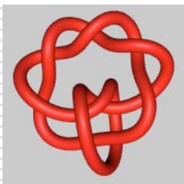
7_1



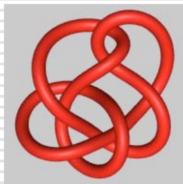
8_{19} =T(4,3)



9_1

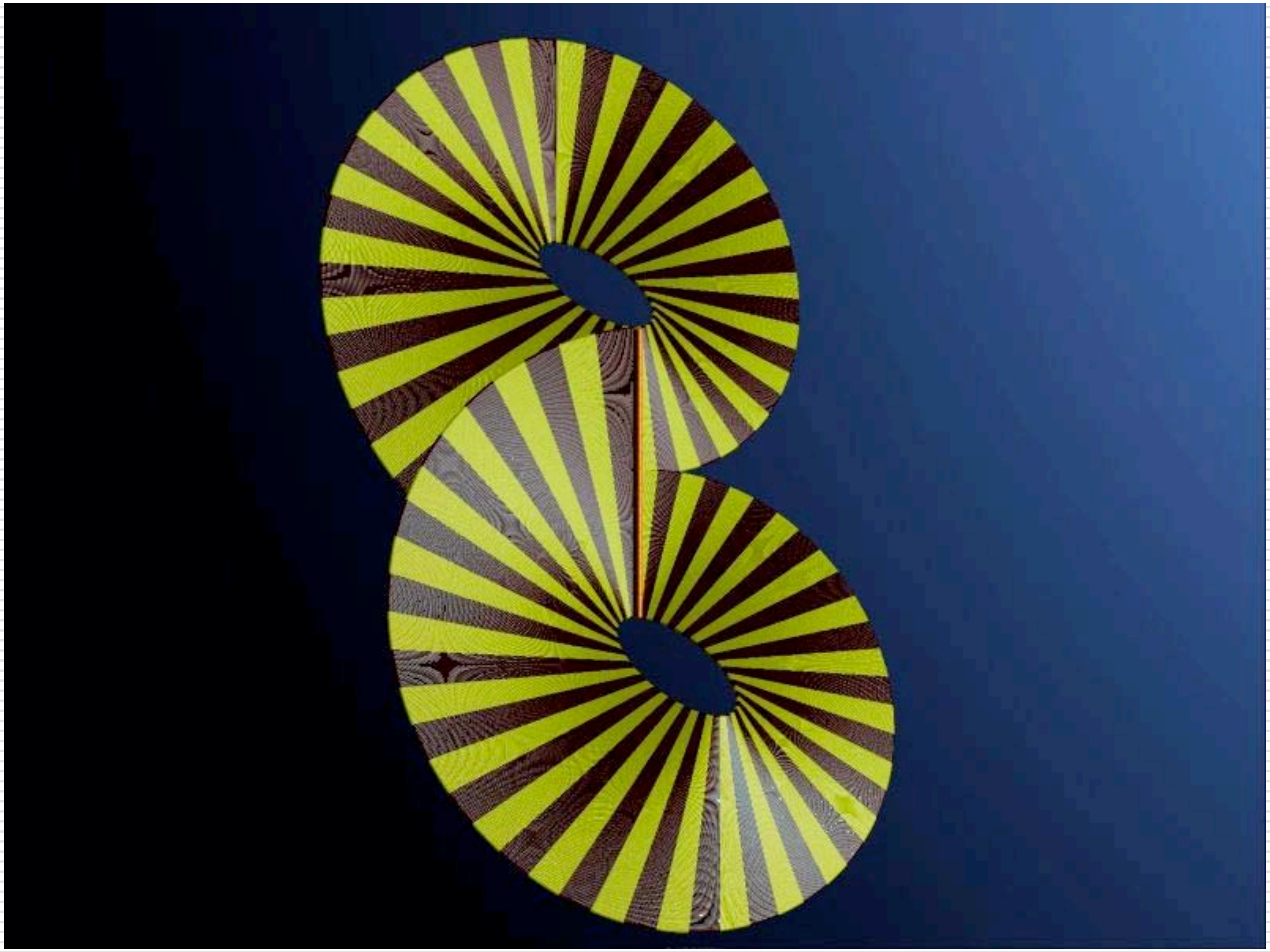


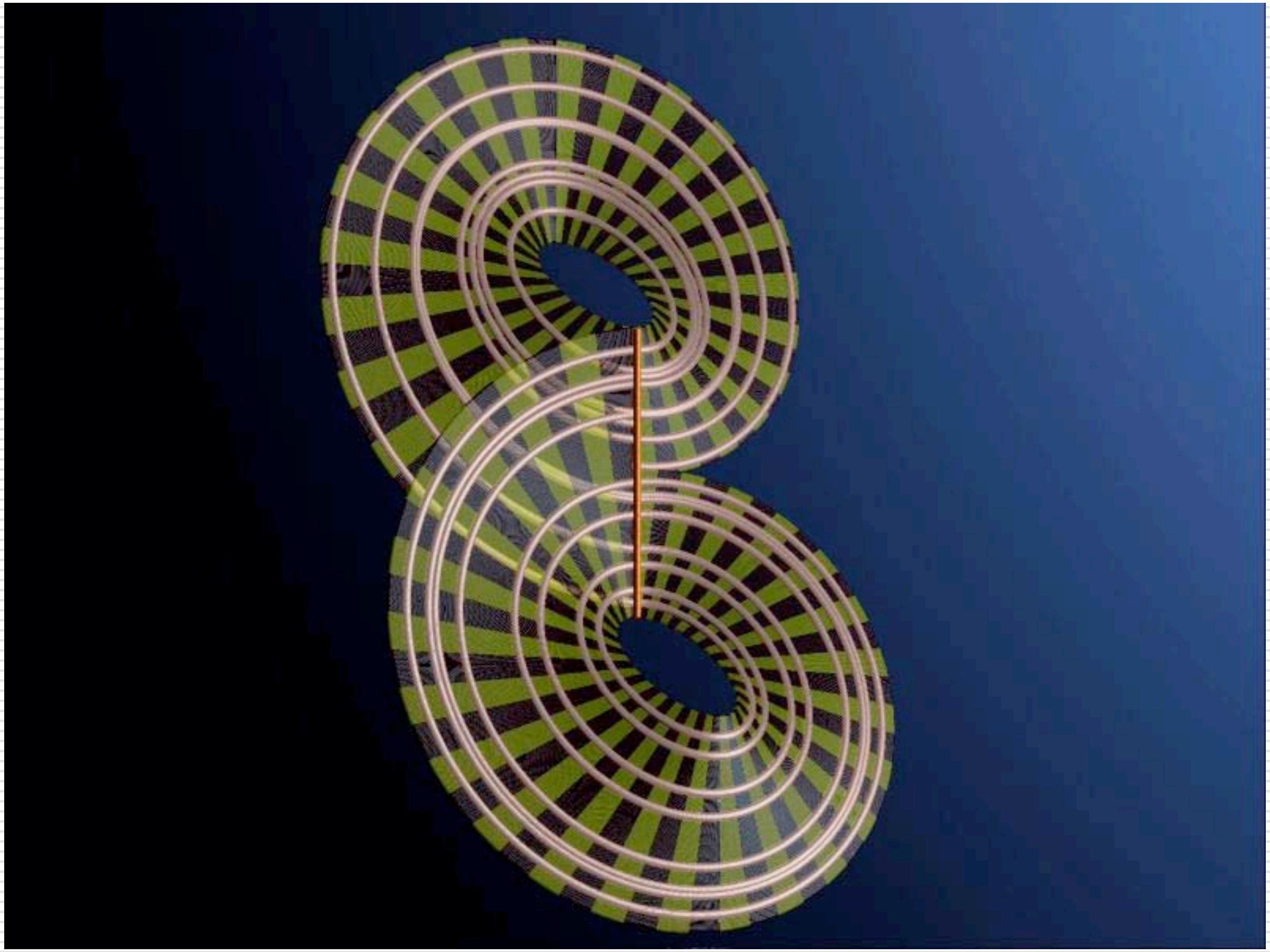
10_{124}=T(5,3)

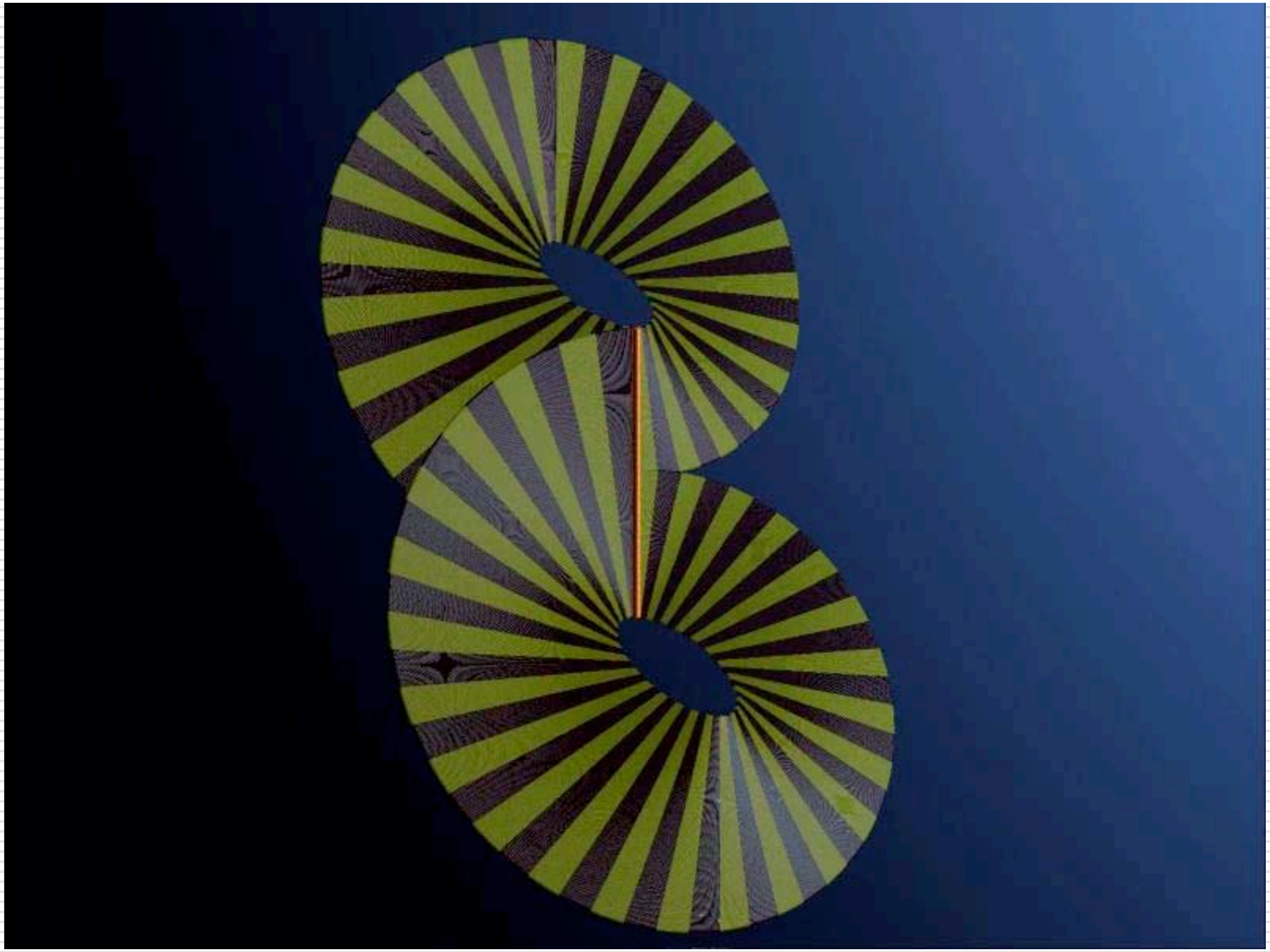


10_{132}

4_1 Figure eight

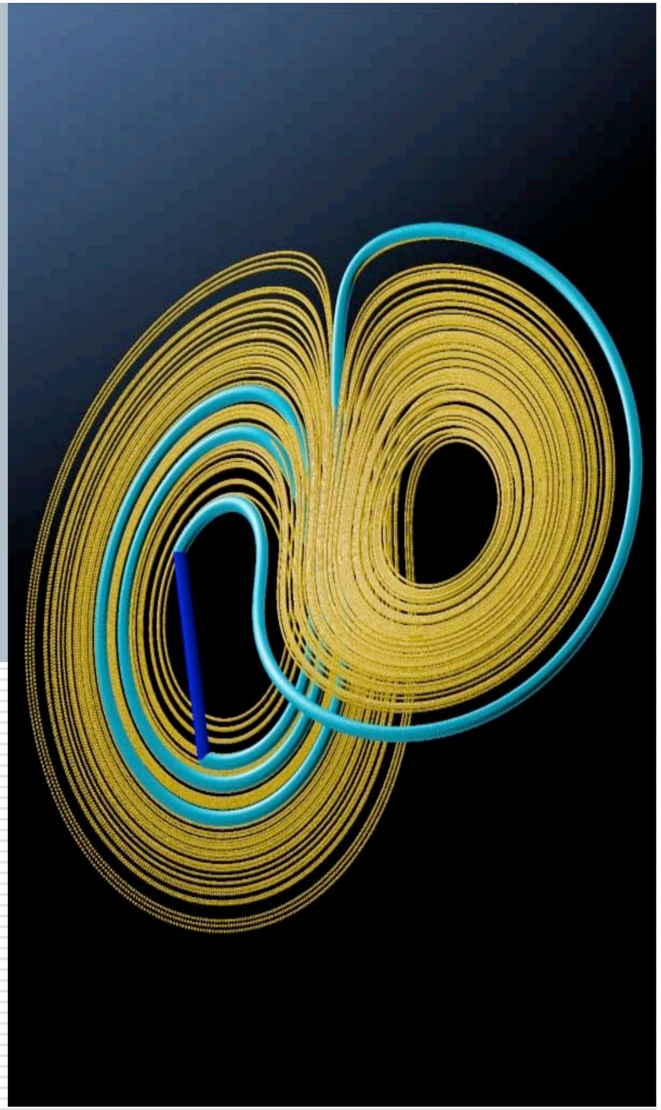






Schwarzman-Sullivan-Thurston etc.

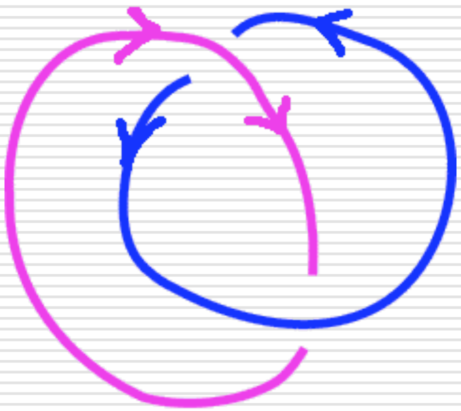
One should think of a measure preserving vector field as « an asymptotic cycle »



$$k(T,x) = \left\{ x \xrightarrow{\phi \text{ orbit}} \phi^T(x) \xrightarrow{\text{segment}} x \right\}$$

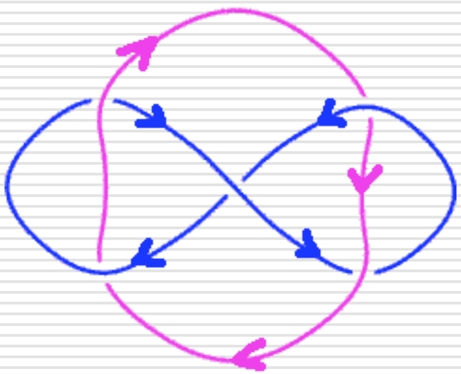
$$Flow \approx \text{"}\lim_{T\to\infty}\text{"} \int \frac{1}{T} k(T,x)\, d\mu(x)$$

Example : helicity as asymptotic linking number



Linking # = +1

+1



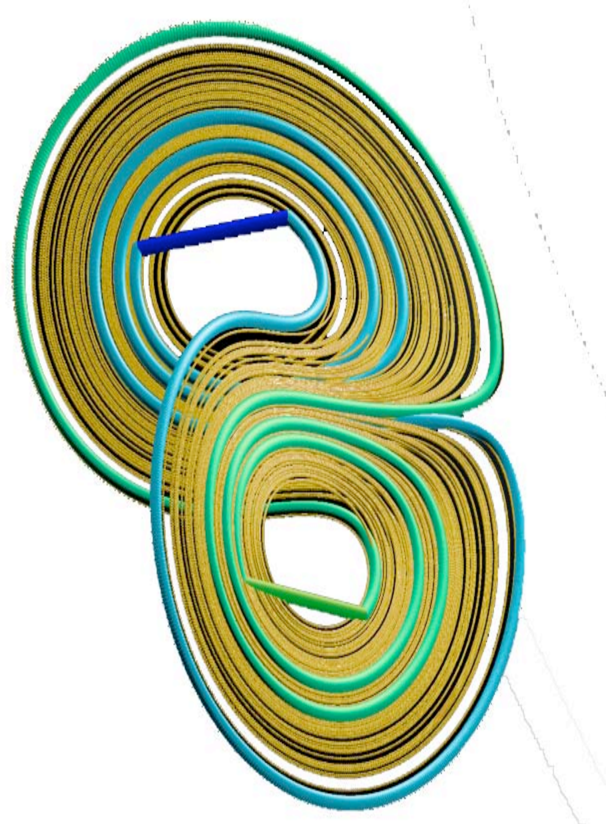
Linking # = 0

-1

Example : helicity as asymptotic linking number

Theorem (Arnold) *Consider a flow in a bounded domain in* $\mathbf{R}^3$ *preserving an ergodic probability measure* μ *(not concentrated on a periodic orbit). Then, for* μ*-almost every pair of points* x_1, x_2*, the following limit exists and is independent of* x_1, x_2

$$Helicity = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} Link(k(T_1, x_1), k(T_2, x_2))$$



Open question (Arnold):

Is ***Helicity*** a topogical invariant ?

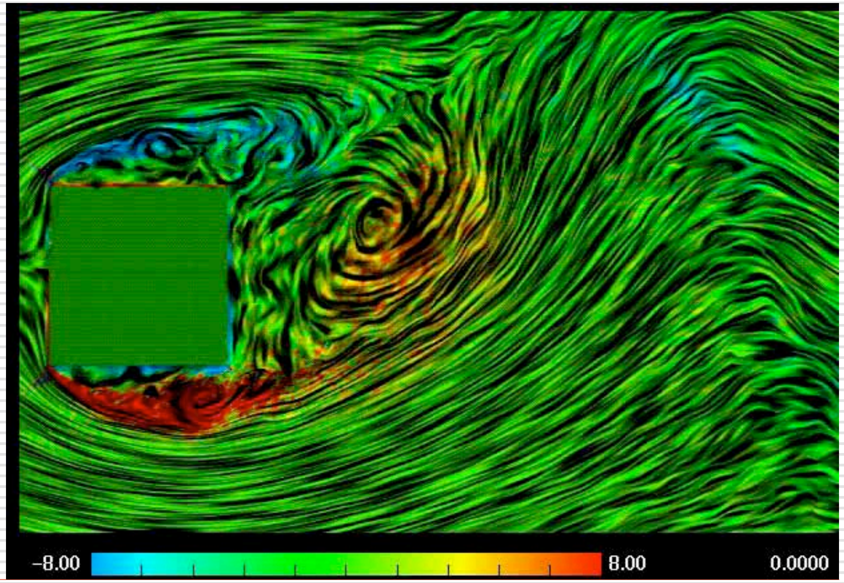
Let ϕ_1^t *and* ϕ_2^t *be two smooth flows preserving* μ_1 *and* μ_1.

Assume there is an orientation preserving homeomorphism

h such that $h \circ \phi_1^t = \phi_2^t \circ h$ *and* $h_* \mu_1 = \mu_2$.

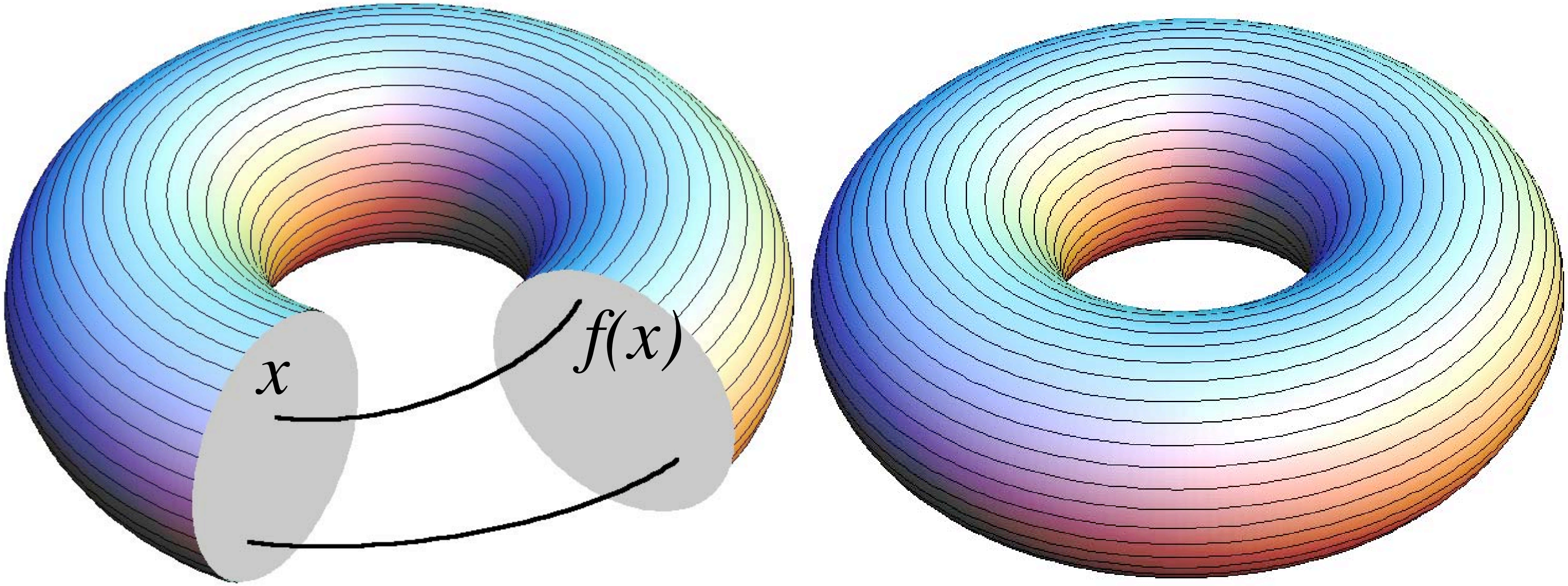
Does it follow that

$$Helicity(\phi_1^t) = Helicity(\phi_2^t)\ ?$$



http://www.math.rug.nl/~veldman/movies/dns-large.mpg

Suspension: an area preserving diffeomorphim of the disk defines a volume preserving vector field in a solid torus.



Invariants on Diff ($\mathbf{D}^2$,*area*)?

Theorem (Calabi): *There is a non trivial homomorphism*

$$Cal : \mathrm{Diff}^{\infty}(\mathrm{D}^2, \partial \mathrm{D}^2, area) \rightarrow \mathbf{R}$$

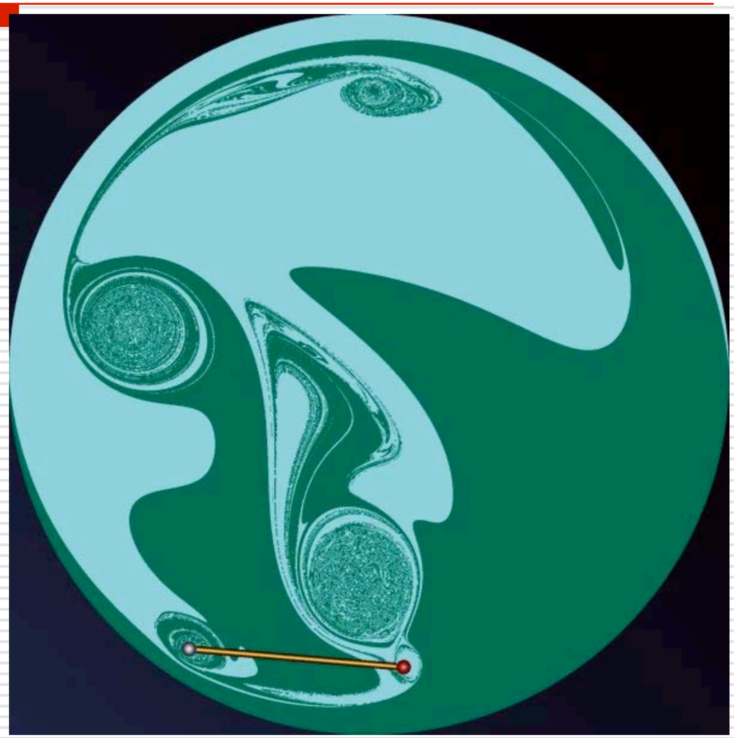
Theorem (Banyaga): *The Kernel of Cal is a simple group.*

Theorem (Gambaudo-G): $Cal(f) = Helicity(\mathrm{Suspension}\, f)$

Theorem (Gambaudo-G): *Cal is a topological invariant.*

Corollary : *Helicity is a topological invariant for those flows which are suspensions.*

A definition of Calabi's invariant (Fathi)



$$f \in \mathrm{Diff}^{\infty}(D^2, \partial D^2, area)$$

$$f_t \quad (t \in [0,1])$$

$$f_0 = Id \qquad f_1 = f$$

$$Cal(f) = \iint Var_{t=0}^{t=1} \, Arg\big(f_t(x) - f_t(y)\big)\, dx\, dy$$

Open question (Mather):

Is the group $\mathrm{Homeo}(\mathbf{D}^2, \partial\,\mathbf{D}^2, \mathit{area})$ *a simple group?*

- Can one extend *Cal* to homeomorphisms?

- Good candidate for a normal subgroup:
the group of « hameomorphisms » (Oh). Is it non trivial?

More invariants on the group Diff($\mathbf{D}^2$, $\partial\,\mathbf{D}^2$, area)?

No homomorphism besides Calabi's…

Quasimorphisms $\chi : \Gamma \to \mathbf{R}$ $\qquad \left|\chi(\gamma_1\gamma_2) - \chi(\gamma_1) - \chi(\gamma_2)\right| \leq Const$

Homogeneous if $\chi(\gamma^n) = n\chi(\gamma)$

Γ non abelian free group

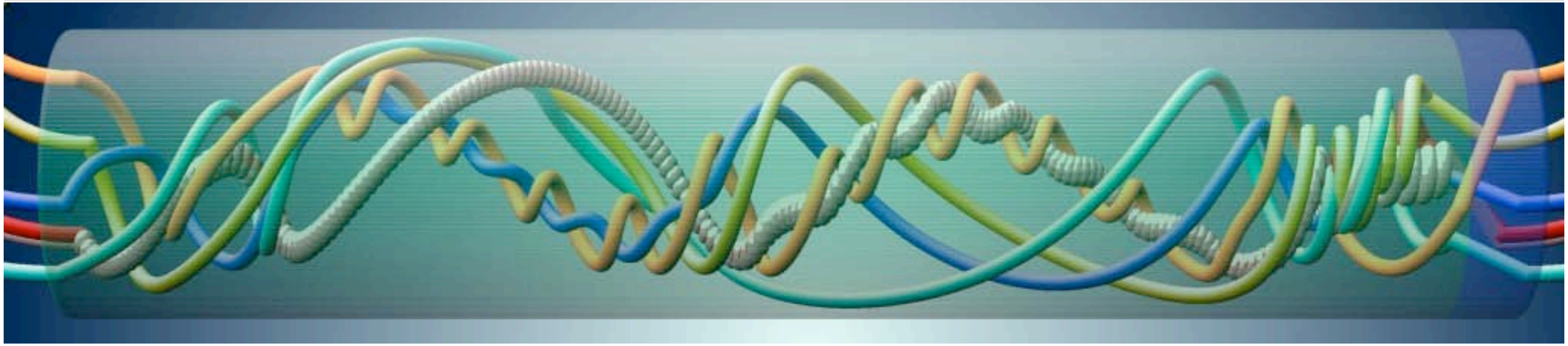
Abelian groups
or
*SL(n,**Z**) for n>2)*

Many non trivial homogeneous quasimorphisms (Gromov, …)

No non trivial (Trauber, Burger-Monod)

Theorem (Gambaudo-G) *The vector space of homogeneous quasimorphisms on* $\mathrm{Diff}(\mathbf{D}^2, \partial\mathbf{D}^2, \mathit{area})$ *is infinite dimensional.*

One idea: use **braids** and quasimorphisms on braid groups.



$$(x_1, x_2, \ldots, x_n) \mapsto b(x_1, x_2, \ldots, x_n) \in \mathrm{B}_n \xrightarrow{\chi} \mathbf{R}$$

Average over *n*-tuples of points in the disk

Example: *n=2*. Calabi homomorphism.

More quasimorphims…

Theorem (Entov-Polterovich) :
There exists a « Calabi quasimorphism » $\chi : \mathrm{Diff}_0(\mathbf{S}^2, area) \to \mathbf{R}$
such that $\chi(f) = Cal(f_{|D})$
when the support of f is in a disc D with area < 1/2 area$(\mathbf{S}^2)$.

Theorem (Py) :
If $\boldsymbol{\Sigma}$ *is a compact orientable surface of genus g > 0,*
there is a « Calabi quasimorphism » $\chi : \mathrm{H}am(\Sigma, area) \to \mathbf{R}$
such that $\chi(f) = Cal(f_{|D})$
when the support of f is contained in some disc $D \subset \boldsymbol{\Sigma}$.

Questions:

- *Can one define similar invariants for volume preserving flows in 3-space, in the spirit of « **Cal**(f) = Helicity(suspension f) »?*

- *Does that produce topological invariants for smooth volume preserving flows?*

- *Higher dimensional topological invariants in symplectic dynamics?*

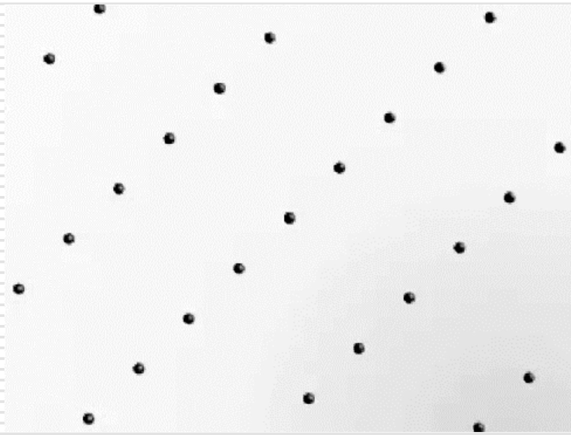
- etc.

Example : *the modular flow on* SL(2,**R**)/SL(2,**Z**)

- Space of lattices in $\mathbf{R}^2$ of area 1

$$\Lambda \approx \mathbf{Z}^2 \subset \mathbf{R}^2$$

$$area(\mathbf{R}^2 / \Lambda) = 1$$

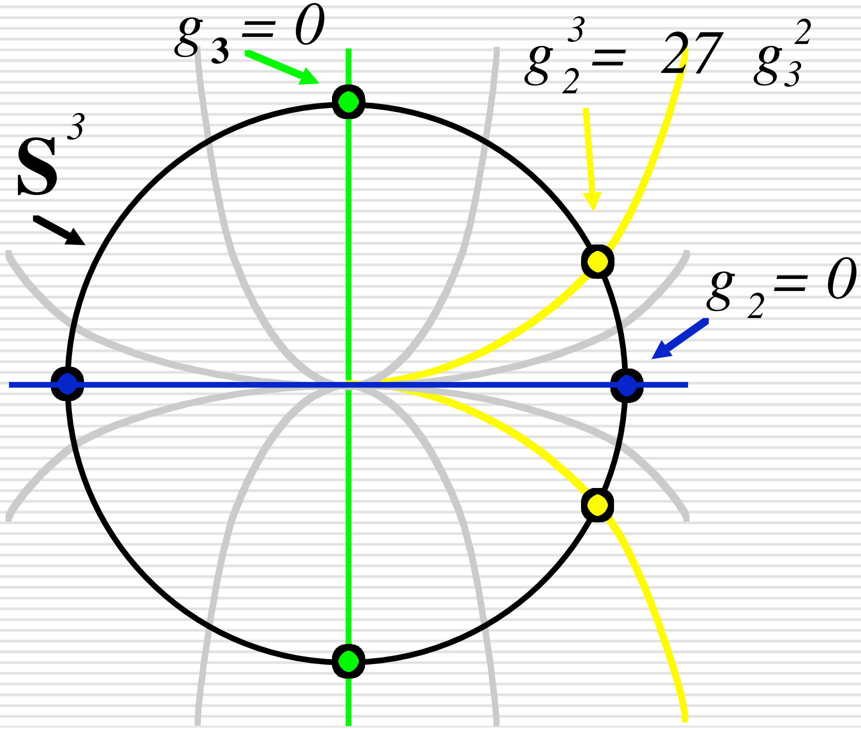


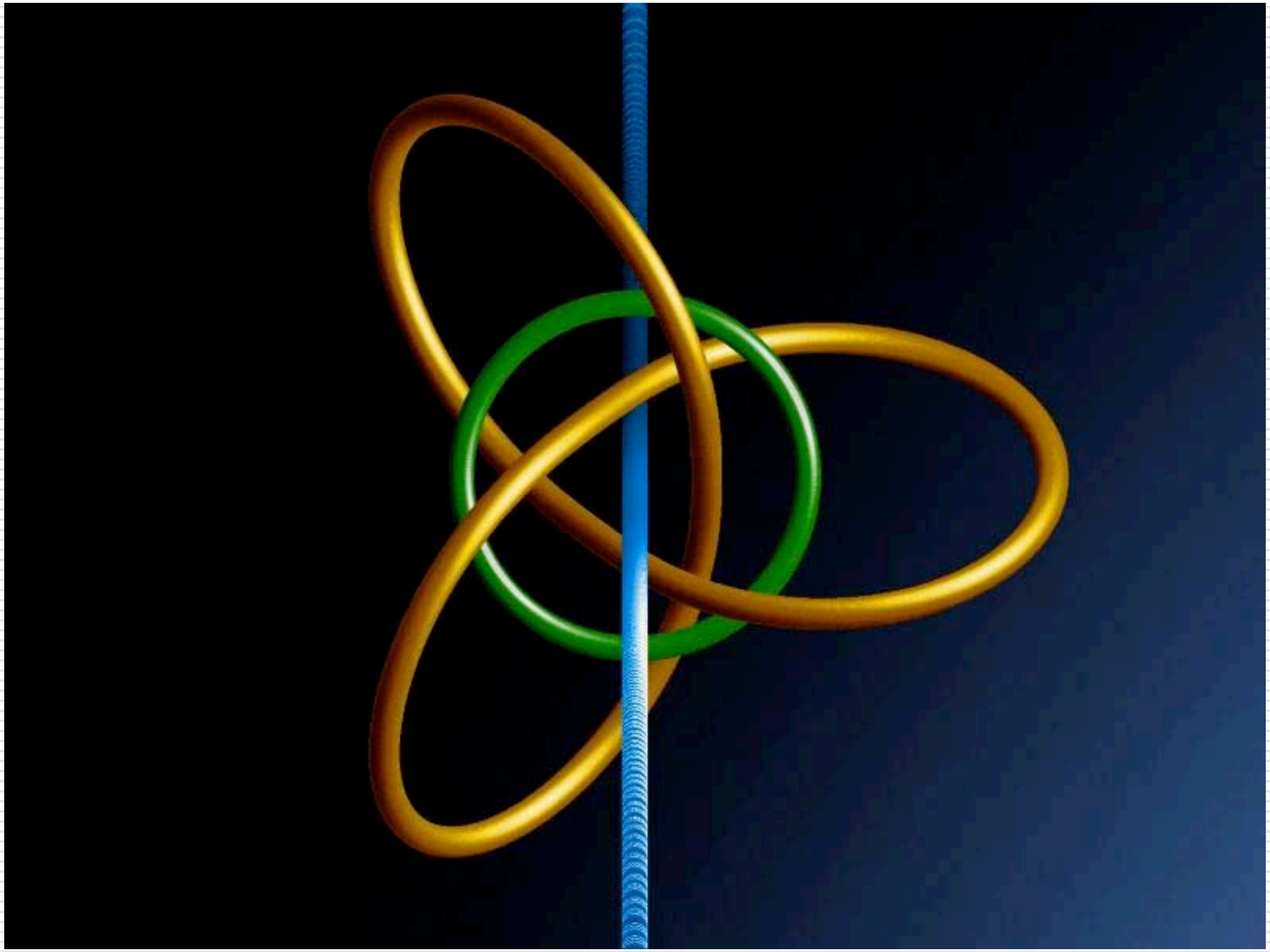
Topology: SL(2,**R**)/SL(2,**Z**) is homeomorphic to the complement of the trefoil knot in the 3-sphere

$$g_2(\Lambda) = 60\sum_{\omega\in\Lambda-\{0\}} \omega^{-4} \in \mathbf{C} \qquad g_3(\Lambda) = 140\sum_{\omega\in\Lambda-\{0\}} \omega^{-6} \in \mathbf{C}$$

$$\Lambda \subset \mathbf{R}^2 \approx \mathbf{C} \mapsto \left(g_2(\Lambda), g_3(\Lambda)\right) \in \mathbf{C}^2 - \left\{g_2{}^3 = 27 g_3{}^2\right\}$$

$$area(\mathbf{R}^2/\Lambda) = 1$$





Dynamics

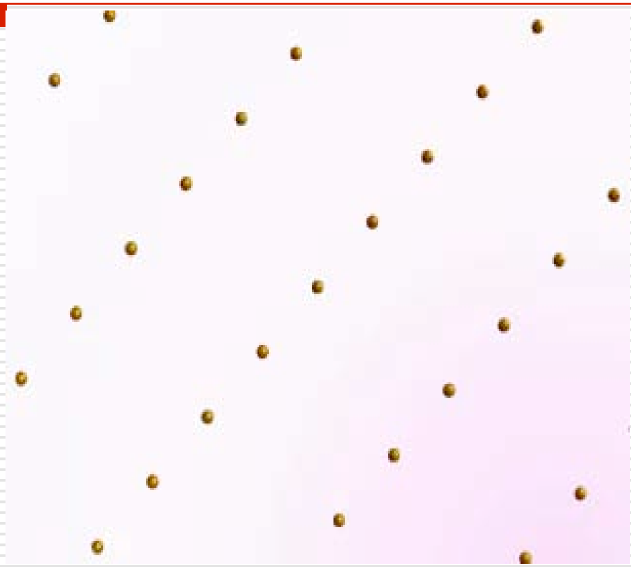
On lattices $\phi^t(\Lambda) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}(\Lambda)$

Periodic orbits

$$A \in SL(2,\mathbf{Z}) \qquad A(\mathbf{Z}^2) = \mathbf{Z}^2$$

$$PAP^{-1} = \pm\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad \Lambda = P(\mathbf{Z}^2)$$

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}(\Lambda) = \Lambda$$



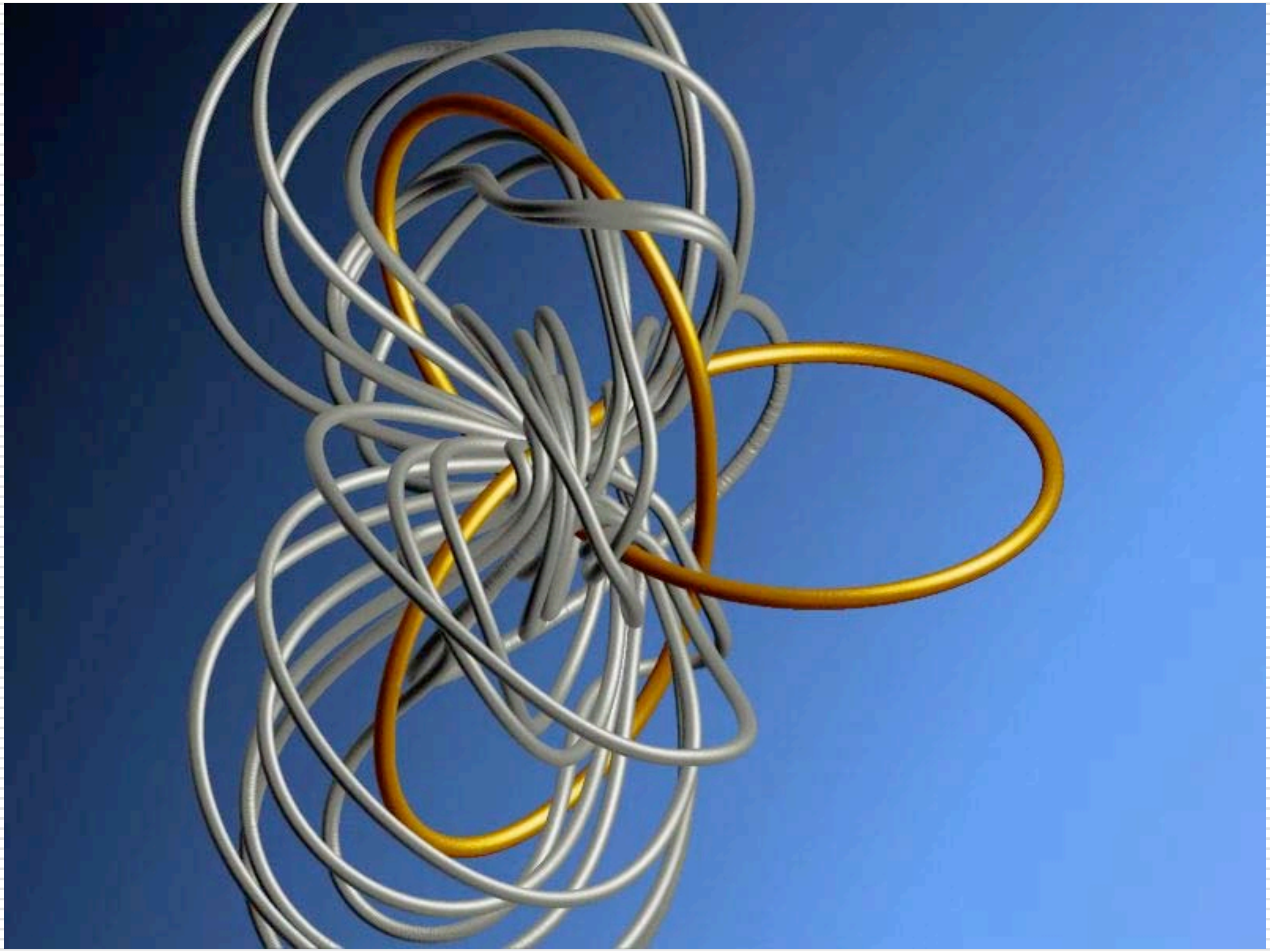
- Conjugacy classes of hyperbolic elements in PSL(2,**Z**)
- Closed geodesics on the modular surface D/PSL(2,**Z**)
- Ideal classes in quadratic fields
- Indefinite integral quadratic forms in two variables.
- Continuous fractions etc.

Each hyperbolic matrix A in PSL(2,**Z**) defines a periodic orbit in SL(2,**R**)/SL(2,**Z**), hence a closed curve $\boldsymbol{k}_{\mathbf{A}}$ in the complement of the trefoil knot.

Questions :

1) What kind of knots are the « modular knots » $\boldsymbol{k}_{\mathbf{A}}$?

2) Compute the linking number between $\boldsymbol{k}_{\mathbf{A}}$ and the trefoil.



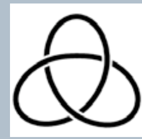
Theorem: *The linking number between* $\boldsymbol{k_A}$ *and the trefoil is equal to* $\mathfrak{R}$(A) *where* $\mathfrak{R}$ *is the « Rademacher function ».*

Dedekind eta-function : $\eta(\tau) = \exp(i\pi\tau/12)\prod_{n\geq 1}\left(1-\exp(2i\pi n\tau)\right) \quad ; \quad \Im(\tau) > 0$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^{24} = \eta(\tau)^{24}(c\tau+d)^{12} \quad ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$$

$$24(\log\eta)\left(\frac{a\tau+b}{c\tau+d}\right) = 24(\log\eta)(\tau) + 6\log(-(c\tau+d)^2) + 2i\pi\mathfrak{R}(A)$$

$\mathfrak{R}:\ SL(2,\mathbf{Z}) \to \mathbf{Z}$ This is a *quasimorphism*.

« Proof » that $\mathfrak{R}(A) = \mathit{Linking}(k_A,$  $)$

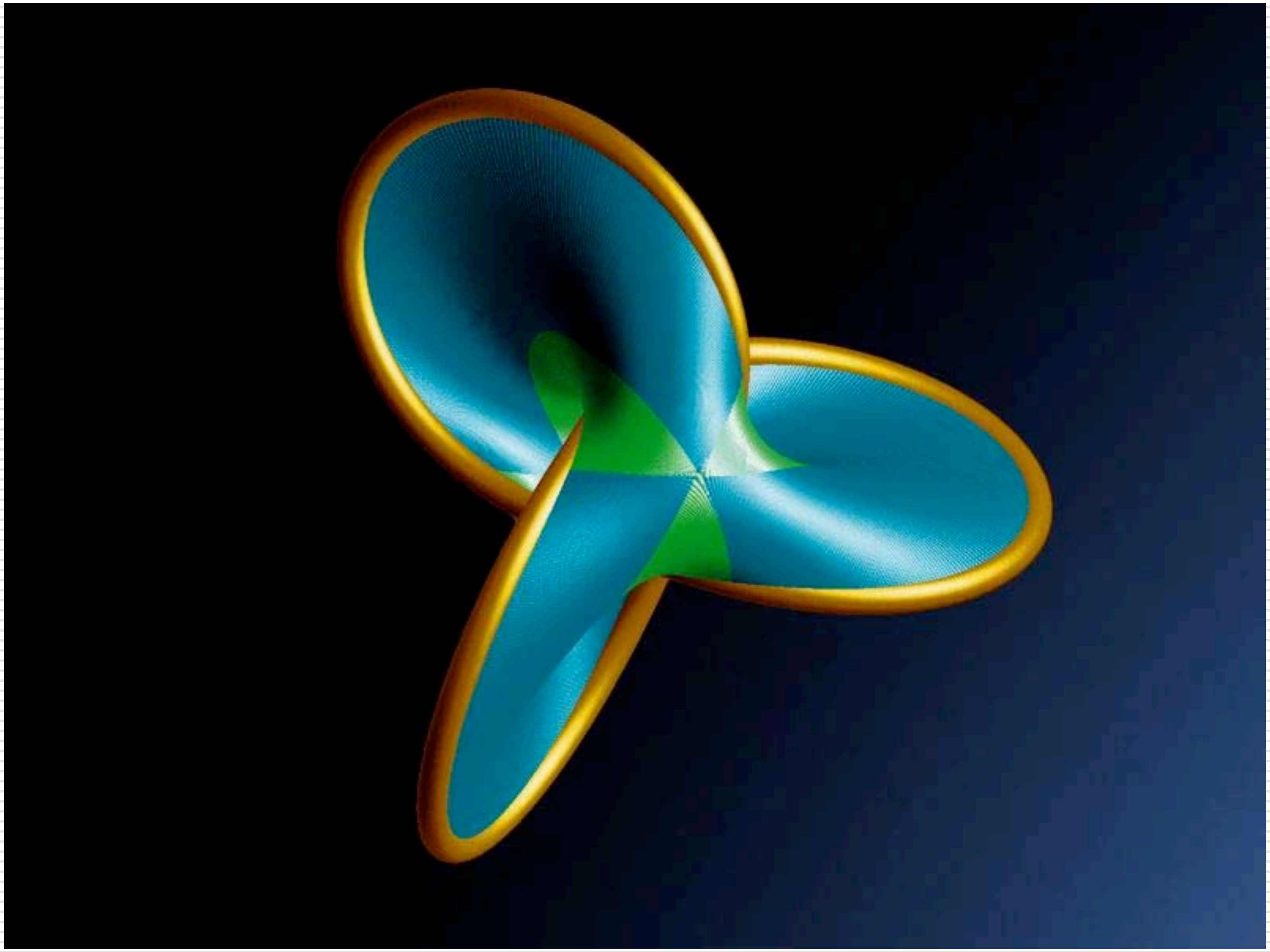
Jacobi proved that

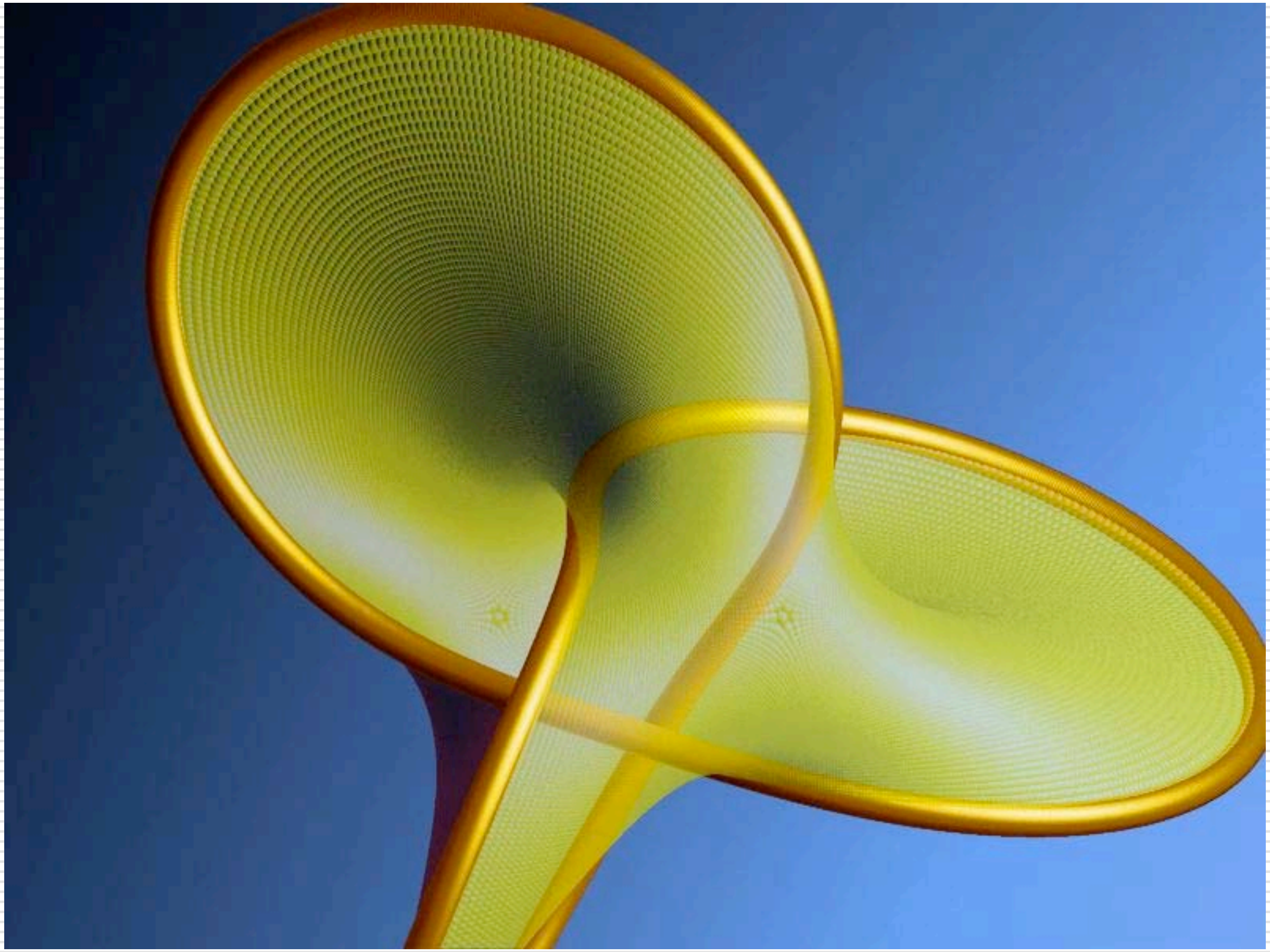
$$(g_2^3 - 27g_3^2)(\mathbf{Z} + \tau\mathbf{Z}) = (2\pi)^{12}\eta(\tau)^{24}$$

$g_2^3 - 27g_3^2 = 0$ is the trefoil knot

$$\log(z) = \log|z| + i\ \mathrm{Arg}(z)$$

$(g_2^3 - 27g_3^2)(\Lambda) \in \mathbf{R}^+$ is a Seifert surface



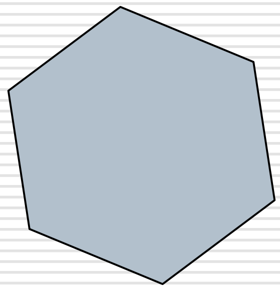


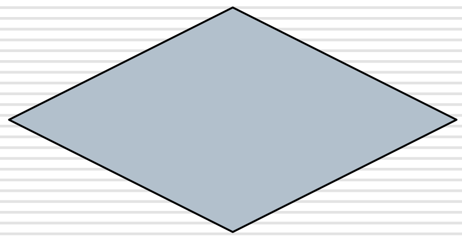
Theorem:

Modular knots (and links) are the same as Lorenz knots (and links)

Step 1: find some template inside SL(2,**R**)/SL(2,**Z**) which looks like the Lorenz template.

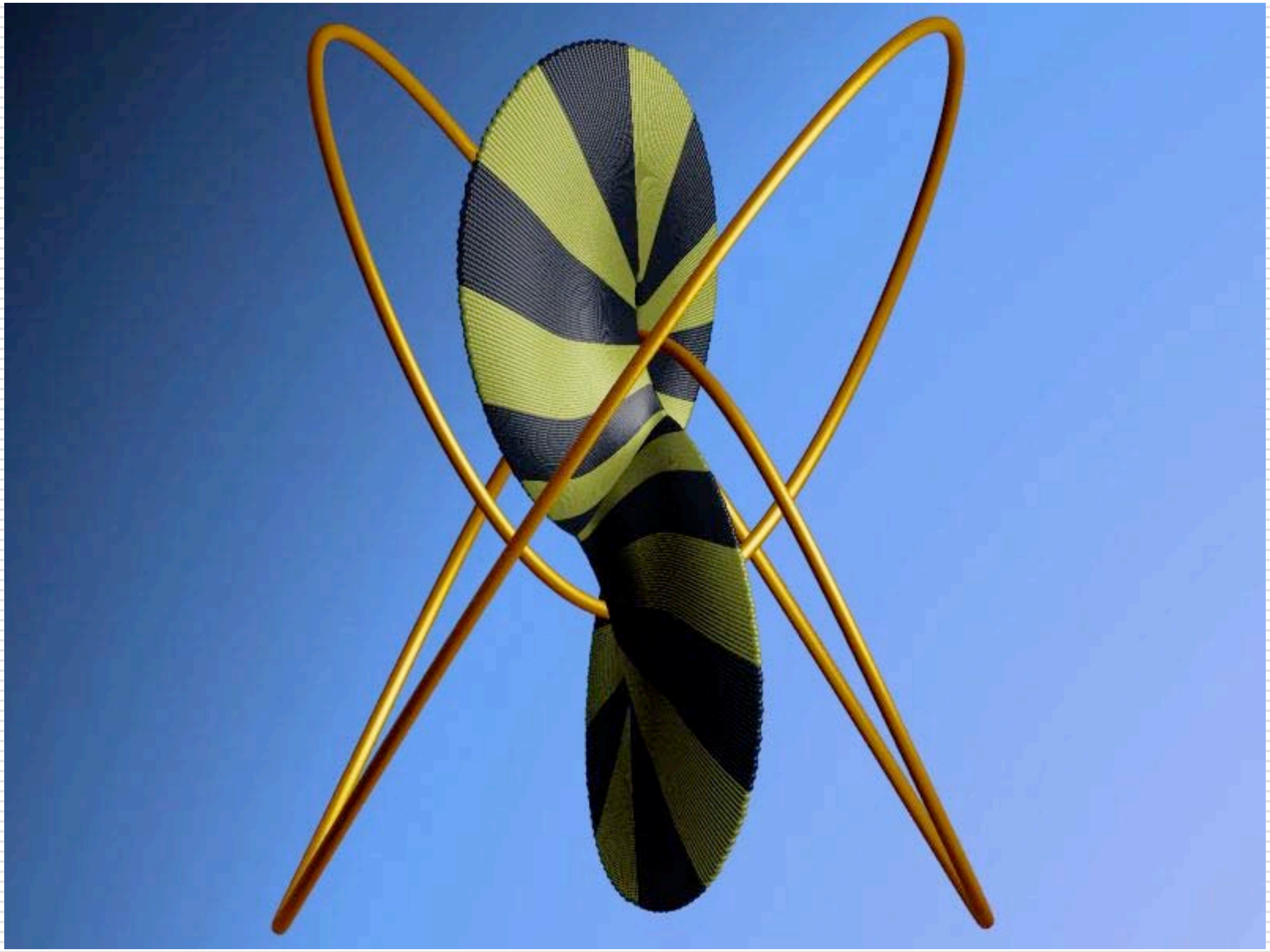
Look at « regular hexagonal lattices »



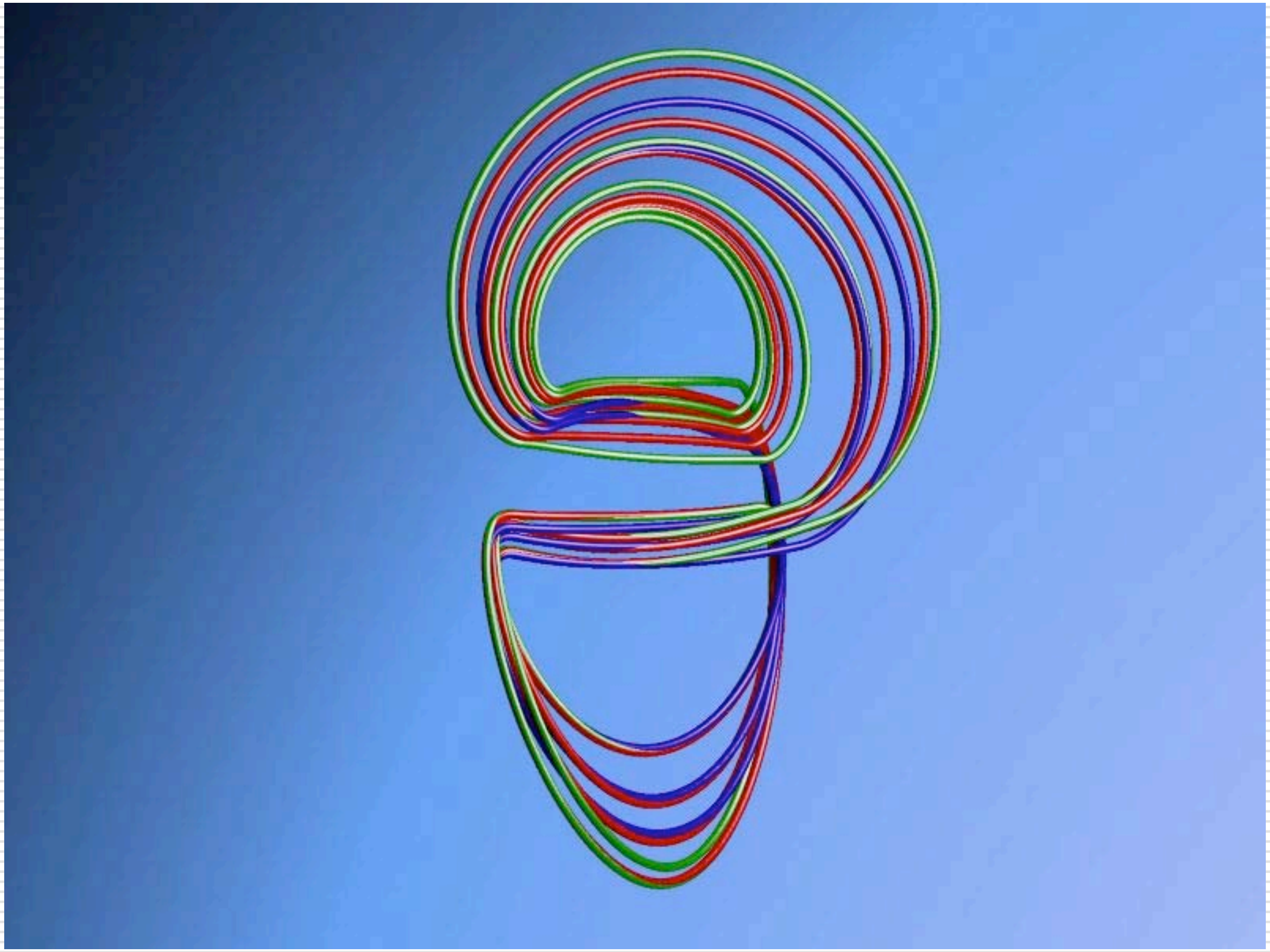


and lattices with horizontal rhombuses as fundamental domain with angle between 60 and 120 degrees.

Make it thicker by pushing along the unstable direction.



Step 2 : deform lattices to make them approach the Lorenz template.



Further developments?

•From modular dynamics to Lorenz dynamics and vice versa?

• For instance, « modular explanation » of the fact that all Lorenz links are fibered?

Many thanks to Jos Leys !

Mathematical Imagery : http://www.josleys.com/

« A mathematical theory is not to be considered complete until you made it so clear that you can explain it to the man you meet on the street »

« For what is clear and easily comprehended attracts and the complicated repels us »

D. Hilbert